69. *Fluctuation of Spectra in Random Media*

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§1. **Introduction.** The Lenz shift phenomena were studied by various authors by various methods. See Kac [5] Huruslov-Marchenko [4], Rauch-Taylor [13], Papanicolaou-Varadhan [14], Ozawa [8], [9], [10], [12], Chavel-Feldman [1]. In [3], Figari-Orlandi-Teta gave fluctuation result for the Lenz shift phenomena by developing the method of [8]. In [8], perturbative calculus using the Green function was offered. It turned out to be strong enough to consider fluctuation of spectra.

In the present note we give Theorem 1 on the Lenz shift.

We consider a bounded domain Ω in $\boldsymbol{R}^3$ with smooth boundary γ. We put $B(\varepsilon; w)=\{x\in \boldsymbol{R}^3; |x-w|<\varepsilon\}$. Fix $\beta\geq 1$. Let

$$0<\mu_1(\varepsilon; w(m))\leq \mu_2(\varepsilon; w(m))\leq \cdots$$

be the eigenvalues of $-\Delta$ $(=-\mathrm{div}\ \mathrm{grad})$ in the set $\Omega_{\varepsilon, w(m)}=\Omega\backslash\overline{\bigcup_{i=1}^{\tilde{m}} B(\varepsilon; w_i^{(m)})}$ under the Dirichlet condition on its boundary. Here $\tilde{m}$ denotes the largest integer which does not exceed m^β and $w(m)$ denotes the set of $\tilde{m}$-points $\{w_i^{(m)}\}_{i=1}^{\tilde{m}}\in\Omega^{\tilde{m}}$. Let $V(x)>0$ be C^0-class function on Ω satisfying

$$\int_\Omega V(x)dx=1.$$

We consider Ω as the probability space with a probability density $V(x)$. Let $\Omega^{\tilde{m}}=\prod_{i=1}^{\tilde{m}}\Omega$ be the probability space with the product measure. Fix $\alpha>0$. Then, $\mu_j(\alpha/m; w(m))$ is a random variable on $\Omega^{\tilde{m}}$. Our aim is to know a precise asymptotic behaviour of $\mu_j(\alpha/m; w(m))$ as $m\to\infty$.

Main result is the following:

Theorem 1. *Fix j. Assume $\beta\in[1, 12/11)$, $V(x)=1/|\Omega|$ (const.). Assume that the j-th eigenvalue μ_j of the Laplacian in Ω under the Dirichlet condition on γ is simple. Then, the random variable*

$$m^{1-(\beta/2)}((\mu_j(\alpha/m; w(m))-(\mu_j+4\pi\alpha m^{\beta-1}|\Omega|^{-1}))) \tag{1}$$

tends in distribution to Gaussian random variable Π_j of mean $E(\Pi_j)=0$ and variance

$$E(\Pi_j^2)=4\pi\alpha\left(\int_\Omega \varphi_j(x)^4|\Omega|^{-1}dx-\left(\int_\Omega \varphi_j(x)^2|\Omega|^{-1}dx\right)^2\right)$$

as m tends to ∞. Here φ_j is the normalized eigenfunction associated with μ_j.

Remark. Figari-Orlandi-Teta's result (see [3]) is the case $\beta=1$. As a corollary of Theorem 1, we have

$$\lim_{m\to\infty} \boldsymbol{P}(w(m)\in\Omega^{\tilde{m}}; (1)\leq c)=\frac{1}{\sqrt{2\pi v}}\int_{-\infty}^{c} e^{-t^2/2v}dt,$$

where $v=E(\Pi_j^2)$.

We make a comment on the Lenz shift. The Lenz shift can be stated as the following: In this case, $\beta=1$.

(2) $$\mu_j(\alpha/m\,;w(m))-(\mu_j+4\pi\alpha|\Omega|^{-1})\to 0$$

in probability. The formula (2) corresponds to a law of large numbers and Theorem 1 corresponds to the central limit theorem.

For other related topics, the readers may be referred to Cioranescu-Murat [2], Ozawa [6], [7], [11].

§2. Sketch of our proof of Theorem 1. Let $\mathcal{O}_1(m)$ be a subset of $w(m)\in\Omega^{\tilde{m}}$ satisfying the following:

$\mathcal{O}_1(m)$: Take an arbitrary open ball K of radius $m^{-\beta/3}$ in $\boldsymbol{R}^3$. Then, the number of balls such that $\overline{\mathrm{ball}}\cap K\neq\emptyset$ is at most $(\log m)^2$. We have

(3) $$\lim_{m\to\infty}\boldsymbol{P}(w(m)\in\Omega^{\tilde{m}}\,;\mathcal{O}_1(m)\text{ holds})=1.$$

Owing to (3), we can restrict ourselves to $\mathcal{O}_1(m)$ to prove Theorem 1. By simple consideration on configuration $w(m)$ of the centers of balls, we see that there exists exactly one large connected component ω of $\Omega_{\alpha/m,w(m)}$ and we see that components other than ω are negligible to consider $\mu_j(\alpha/m\,;w(m))$ as $m\to\infty$.

Put $\lambda=Tm^{\beta-1}$ (T=constant). Let $G(x,y)$ be the Green function of $-\Delta+\lambda$ in Ω under the Dirichlet condition on γ. Let $G(x,y\,;w(m))$ be the Green function of $-\Delta+\lambda$ in ω under the Dirichlet condition on $\partial\omega$.

We introduce the following integral kernel $h(x,y\,;w(m))$: We abbreviate $w_i^{(m)}$ as w_i, and $G(w_i,w_j)$ as G_{ij}. Put $\lambda^{1/2}m^{-1}=\tau$, and $m^*=(\log m)^2$.

$$h(x,y\,;w(m))=G(x,y)+(-4\pi\alpha/m)e^{\tau}\textstyle\sum_{i=1}^{\tilde{m}}G(x,w_i)G(w_i,y)$$
$$+\textstyle\sum_{s=1}^{m^*}(-4\pi\alpha/m)^s e^{s\tau}\sum_{(s)}G(x,w_{i_1})G_{i_1i_2}\cdots G_{i_{s-1}i_s}G(w_{i_s},y).$$

Here the indices in $\Sigma_{(s)}$ run over all $1\le i_1,\cdots,i_s\le\tilde{m}$ such that $i_\nu\neq i_\mu$ when $\nu\neq\mu$. The sum $\Sigma_{(s)}$ is called self-avoiding sum.

Put

$$(\boldsymbol{H}_{w(m)}f)(x)=\int_\omega h(x,y\,;w(m))f(y)dy,\qquad x\in\omega$$

and

$$(\tilde{\boldsymbol{H}}_{w(m)}g)(x)=\int_\Omega h(x,y\,;w(m))g(y)dy,\qquad x\in\Omega.$$

Let $\boldsymbol{G}_{w(m)}$ be the bounded linear operator on $L^2(\omega)$ defined by

$$(\boldsymbol{G}_{w(m)}g)(x)=\int_\omega G(x,y\,;w(m))g(y)dy.$$

Let A denote the Green operator of $-\Delta+4\pi\alpha|\Omega|^{-1}m^{\beta-1}+\lambda$ in Ω under the Dirichlet condition on γ.

Main line of our proof of Theorem 1 is comparison of $\boldsymbol{G}_{w(m)}$ with $\boldsymbol{A}$.

Let $\mathcal{O}_2(m)$ be a set of $w(m)\in\Omega^{\tilde{m}}$ satisfying the following:

$\mathcal{O}_2(m)$: There exists a constant C independent of m such that

$$\textstyle\sum_{(s+1)}|w_{i_1}-w_{i_2}|^{-2+\sigma}\exp(-\lambda^{1/2}|w_{i_1}-w_{i_2}|)G_{i_2i_3}\cdots G_{i_si_{s+1}}$$
$$\le C^{s+1}\lambda^{-s+(1/2)(1-\sigma)}m^{\beta(s+1)}(\log m)^3$$

holds for any $1\le s\le(\log m)^2$, $\sigma=0,1$.

We have

(4) $$\lim_{m\to\infty} \boldsymbol{P}(w(m)\in\Omega^{\bar{m}};\ \mathcal{O}_2(m) \text{ holds})=1.$$

We have the following Propositions.

Proposition 1. *Assume that $w(m)$ satisfy $\mathcal{O}_1(m)$ and $\mathcal{O}_2(m)$. Fix an arbitrary $\varepsilon>0$. Then,*

$$\|\boldsymbol{G}_{w(m)}-\boldsymbol{H}_{w(m)}\|_{L^2(\omega)}\leq Cm^{\beta-2+\varepsilon}$$

holds for a constant C independent of m.

Proposition 2. *Assume that $w(m)$ satisfy $\mathcal{O}_1(m)$ and $\mathcal{O}_2(m)$. Fix an arbitrary $\varepsilon>0$. Then,*

$$\|\tilde{\boldsymbol{H}}_{w(m)}-\chi_\omega\tilde{\boldsymbol{H}}_{w(m)}\chi_\omega\|_{L^2(\Omega)}\leq CD(m)m^\varepsilon,$$

where χ_ω is the characteristic function of ω. Here

$$D(m)=m^{(\beta-3)/3}+m^{(3\beta-5)/2}+m^{2(\beta-3)/3}+m^{\beta-2}+m^{(\beta-9)/6}.$$

Proposition 3. *Fix an arbitrary positive $\varepsilon>0$. Then, the measure of the set $w(m)$ satisfying*

$$\|\tilde{\boldsymbol{H}}_{w(m)}-\boldsymbol{A}\|_{L^2(\Omega)}\leq m^{\varepsilon-(\beta/2)}F(m)$$

tends to 1 as $m\to\infty$. Here $F(m)=m^{1-(3/2)\beta}+m^{(1-3\beta)/4}+m^{-1/2}+m^{-(\beta-1)/4}$.

We know that $\boldsymbol{G}_{w(m)}-\boldsymbol{H}_{w(m)}$, $\tilde{\boldsymbol{H}}_{w(m)}-\chi_\omega\tilde{\boldsymbol{H}}_{w(m)}\chi_\omega$ is negligible to consider fluctuation of spectra and that fluctuation arises from $\tilde{\boldsymbol{H}}_{w(m)}-\boldsymbol{A}$, if $\beta\in[1, 12/11)$.

Details of this paper will be given in [12].

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