67. On the Generators of Exponentially Bounded C-semigroups

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1. Introduction. Let X be a Banach space, and let $C: X \to X$ be an injective bounded linear operator with dense range. According to Davies and Pang [1], we say that $\{S(t): t \geqq 0\}$ is an *exponentially bounded C-semigroup* if $S(t): X \to X$, $0 \leqq t < \infty$, is a family of bounded linear operators satisfying

(1.1) $S(t+s)C = S(t)S(s)$ for $t, s \geqq 0$, and $S(0) = C$,

(1.2) for every $x \in X$, $S(t)x$ is continuous in $t \geqq 0$,

(1.3) there exist $M \geqq 0$ and real a such that $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$.

For every $t \geqq 0$, let $T(t)$ be the closed linear operator defined by

$$T(t)x = C^{-1}S(t)x \qquad \text{for } x \in D(T(t))$$

with $D(T(t)) = \{x \in X : S(t)x \in R(C)\}$. We define the operator G by

$$D(G) = \{x \in R(C) : \lim_{t \to 0+} (T(t)x - x)/t \text{ exists}\}$$

and

(1.4) $$Gx = \lim_{t \to 0+} (T(t)x - x)/t \qquad \text{for } x \in D(G).$$

For every $\lambda > a$, define the bounded linear operator $L_\lambda : X \to X$ by

(1.5) $$L_\lambda x = \int_0^\infty e^{-\lambda t} S(t)x \, dt \qquad \text{for } x \in X.$$

It is known that L_λ is injective and the closed linear operator Z defined by

(1.6) $$Zx = L_\lambda^{-1}(\lambda L_\lambda - C)x = (\lambda - L_\lambda^{-1}C)x \qquad \text{for } x \in D(Z)$$

with $D(Z) = \{x \in X : Cx \in R(L_\lambda)\}$ is independent of $\lambda > a$. (See [1].) The operator Z will be called the *generator* of $\{S(t): t \geqq 0\}$.

Remark. If $C = I$ (the identity on X), then every exponentially bounded C-semigroup is a C_0-semigroup in the ordinary sense. In this case, (1.1) and (1.2) imply (1.3), and the generator Z coincides with G defined by (1.4) (see [2, 3]). However, these do not hold in general (see [1]).

The purpose of this paper is to prove the following theorems.

Theorem 1. *Let $\{S(t): t \geqq 0\}$ be an exponentially bounded C-semigroup satisfying $\|S(t)\| \leqq Me^{at}$, and let G be the operator defined by* (1.4). *Then G is closable and $\bar{G}$ (the closure of G) is a densely defined linear operator in X satisfying the following conditions*

(a_1) *$\lambda - \bar{G}$ is injective for $\lambda > a$,*

(a_2) *$D((\lambda - \bar{G})^{-n}) \supset R(C)$ for $n \geqq 1$ and $\lambda > a$,*

(a_3) *$\|(\lambda - \bar{G})^{-n}C\| \leqq M/(\lambda - a)^n$ for $n \geqq 1$ and $\lambda > a$,*

(a_4) *$(\lambda - \bar{G})^{-1}Cx = C(\lambda - \bar{G})^{-1}x$ for $x \in D((\lambda - \bar{G})^{-1})$ and $\lambda > a$.*

Theorem 2. *If T is a densely defined closed linear operator in X* *satisfying* (a_1)–(a_4) *in Theorem* 1, *then the operator* $C^{-1}TC$ *with* $D(C^{-1}TC)=\{x\in X: Cx\in D(T)$ *and* $TCx\in R(C)\}$ *is the generator of an exponentially bounded C-semigroup* $\{S(t):t\geqq 0\}$ *satisfying* $\|S(t)\|\leqq Me^{at}$ *for* $t\geqq 0$.

Corollary. *Let Z be the generator of an exponentially bounded C-semigroup* $\{S(t):t\geqq 0\}$ *and let G be the operator defined by* (1.4). *Then we have*

$$Z=C^{-1}\overline{G}C,$$

where $D(C^{-1}\overline{G}C)=\{x\in X: Cx\in D(\overline{G})$ *and* $\overline{G}Cx\in R(C)\}$.

These theorems give a generalization of the Hille-Yosida theorem.

2. Proof of Theorem 1. Throughout this section let Z be the generator of an exponentially bounded C-semigroup $\{S(t):t\geqq 0\}$ satisfying $\|S(t)\|\leqq Me^{at}$, and let G be the operator defined by (1.4).

The following (2.1) and (2.2) have been proved in [1]:

(2.1) $D(G)$ is dense in X and $G\subset Z$.

(2.2) $$(\lambda-G)L_\lambda Cx=C^2x \quad \text{for } x\in X \text{ and } \lambda>a,$$ $$L_\lambda(\lambda-G)x=Cx \quad \text{for } x\in D(G) \text{ and } \lambda>a.$$

Noting that G is closable and $R(C)$ is dense in X, (2.2) implies

(2.3) $$(\lambda-\overline{G})L_\lambda x=Cx \quad \text{for } x\in X \text{ and } \lambda>a,$$ $$L_\lambda(\lambda-\overline{G})x=Cx \quad \text{for } x\in D(\overline{G}) \text{ and } \lambda>a.$$

Proof of Theorem 1. (a_1) and (a_4) follow from (2.3). To prove (a_2) and (a_3), let $\lambda>a$. By (2.3) again, we obtain

(2.4) $$D((\lambda-\overline{G})^{-n})\supset R(C^n) \quad \text{and} \quad (\lambda-\overline{G})^{-n}C^nz=(L_\lambda)^nz$$

for $n\geqq 1$ and $z\in X$. Since

$$(L_\lambda)^nz=\int_0^\infty\cdots\int_0^\infty e^{-\lambda(t_1+\cdots+t_n)}S(t_1+\cdots+t_n)C^{n-1}z\,dt_1\cdots dt_n,$$

$\|(\lambda-\overline{G})^{-n}C^nz\|=\|(L_\lambda)^nz\|\leqq(M/(\lambda-a)^n)\|C^{n-1}z\|$ for $n\geqq 1$ and $z\in X$. So that we have for every $n\geqq 1$

(2.5) $$\|(\lambda-\overline{G})^{-n}Cx\|\leqq(M/(\lambda-a)^n)\|x\| \quad \text{for } x\in R(C^{n-1}).$$

Now, (2.4) and the closedness of $\overline{G}$ imply

(2.6) $$D((\lambda-\overline{G})^{-n})\supset R(C) \quad \text{and}$$ $$(\lambda-\overline{G})^{-n}C \text{ is a bounded linear operator on } X$$

for every $n\geqq 1$. Indeed, by (2.4), (2.6) holds for $n=1$. Suppose that (2.6) is true for $n=k$. Let $x\in X$. Since $R(C^k)$ is dense in X, there are $x_m\in R(C^k)$ such that $x_m\to x$ as $m\to\infty$. Then

$$\lim_{m\to\infty}(\lambda-\overline{G})^{-k}Cx_m=(\lambda-\overline{G})^{-k}Cx$$

and by (2.5)

$$\|(\lambda-\overline{G})^{-(k+1)}Cx_p-(\lambda-\overline{G})^{-(k+1)}Cx_m\|\leqq(M/(\lambda-a)^{k+1})\|x_p-x_m\|\to 0$$

as $p, m\to\infty$ and hence $(\lambda-\overline{G})^{-1}[(\lambda-\overline{G})^{-k}Cx_m]=(\lambda-\overline{G})^{-(k+1)}Cx_m$ is convergent as $m\to\infty$. Since $(\lambda-\overline{G})^{-1}$ is closed we see that $(\lambda-\overline{G})^{-k}Cx\in D((\lambda-\overline{G})^{-1})$ i.e., $Cx\in D((\lambda-\overline{G})^{-(k+1)})$. Therefore $D((\lambda-\overline{G})^{-(k+1)})\supset R(C)$, and $(\lambda-\overline{G})^{-(k+1)}C=(\lambda-\overline{G})^{-1}(\lambda-\overline{G})^{-k}C$ is a closed linear operator on X and then it is bounded on X by the closed graph theorem. So (2.6) holds for $n=k+1$.

Thus (a_2) is satisfied. (a_3) follows from (2.5) and (2.6) since $R(C^{n-1})$ is dense in X for every $n\geqq 1$. Q.E.D.

3. Proof of Theorem 2 and Corollary. Throughout this section T denotes a densely defined closed linear operator in X satisfying (a_1)–(a_4) in Theorem 1. It is easy to see that (a_4) is equivalent to the following

$(a_4)'$ $Cz \in D(T)$ and $TCz = CTz$ for $z \in D(T)$.

For every $\lambda > a$ and $t \geqq 0$, define $S_\lambda(t) : X \to X$ by

$$S_\lambda(t)x = e^{-\lambda t} \sum_{n=0}^{\infty} (t^n \lambda^{2n}/n!)(\lambda - T)^{-n}Cx \qquad \text{for } x \in X.$$

By virtue of [1, Theorem 11], $\lim_{\lambda\to\infty} S_\lambda(t)x$ exists for every $x \in X$ and $t \geqq 0$, and if we define $S(t)$ for $t \geqq 0$ by

$$(3.1) \qquad S(t)x = \lim_{\lambda\to\infty} S_\lambda(t)x \qquad \text{for } x \in X$$

then $\{S(t) : t \geqq 0\}$ is an exponentially bounded C-semigroup satisfying $\|S(t)\| \leqq Me^{at}$ for $t \geqq 0$ and

$$(3.2) \qquad (\lambda - T)^{-1}Cx = \int_0^\infty e^{-\lambda t} S(t)x\,dt \qquad \text{for } x \in X \text{ and } \lambda > a.$$

By $(a_4)'$, $S_\lambda(t)z \in D(T)$ and $TS_\lambda(t)z = S_\lambda(t)Tz$ for $z \in D(T)$, $t \geqq 0$ and $\lambda > a$, and hence the closedness of T implies

$$(3.3) \qquad S(t)z \in D(T) \text{ and } TS(t)z = S(t)Tz \qquad \text{for } z \in D(T) \text{ and } t \geqq 0.$$

Lemma. *If Z is the generator of $\{S(t) : t \geqq 0\}$, then $T \subset Z$ and*

$$Cx \in D(T) \text{ and } TCx = CZx \qquad \text{for } x \in D(Z).$$

Proof. $T \subset Z$ has been proved in [1, Remark after Theorem 13]. Let $\lambda > a$. For $x \in X$, $(\lambda - T)L_\lambda x = Cx$ by (3.2), and $(\lambda - Z)L_\lambda x = Cx$ by (2.3) and $Z \supset \overline{G}$, where G is defined by (1.4). Therefore we have

$$(3.4) \qquad TL_\lambda x = ZL_\lambda x \qquad \text{for } x \in X.$$

Now, let $x \in D(Z)$. By (3.4) and $ZL_\lambda x = L_\lambda Zx$

$$T(\lambda L_\lambda x) = \lambda L_\lambda Zx \qquad \text{for } \lambda > a.$$

Since $\lambda L_\lambda x \to Cx$ and $\lambda L_\lambda Zx \to CZx$ as $\lambda \to \infty$, the closedness of T implies that $Cx \in D(T)$ and $TCx = CZx$. Q.E.D.

Proof of Theorem 2. We want to show that $C^{-1}TC$ is the generator of $\{S(t) : t \geqq 0\}$. Set $T' = C^{-1}TC$. Then $T' \supset T$ by $(a_4)'$, and hence it is easily seen that T' is a densely defined closed linear operator satisfying (a_1)–(a_4). Therefore we can construct an exponentially bounded C-semigroup $\{S(t)' : t \geqq 0\}$ by

$$S(t)'x = \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^{\infty} (t^n \lambda^{2n}/n!)(\lambda - T')^{-n}Cx$$

for $x \in X$ and $t \geqq 0$. Since $(\lambda - T')^{-n}C = (\lambda - T)^{-n}C$ ($n \geqq 0$, $\lambda > a$) by $T' \supset T$, we see that $S(t)' = S(t)$ for $t \geqq 0$. Let Z be the generator of $\{S(t) : t \geqq 0\}$ $(= \{S(t)' : t \geqq 0\})$. By virtue of Lemma, $T' \subset Z$, and if $x \in D(Z)$ then $Cx \in D(T)$ and $TCx = CZx \in R(C)$ i.e., $x \in D(T')$. So that $T' \subset Z$ and $D(Z) \subset D(T')$, and hence $T' = Z$. Q.E.D.

Proof of Corollary. Since $\overline{G}$ is a densely defined closed linear operator in X satisfying (a_1)–(a_4) by Theorem 1, it follows from Theorem 2 that $C^{-1}\overline{G}C$ is the generator of an exponentially bounded C-semigroup $\{S(t)' : t \geqq 0\}$ defined by

$$S(t)'x = \lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^{\infty} (t^n \lambda^{2n}/n!)(\lambda - \overline{G})^{-n}Cx \qquad \text{for } x \in X \text{ and } t \geqq 0.$$

It has been proved in [1, Theorems 10 and 13] that Z is a densely defined closed linear operator in X satisfying (a_1)–(a_4) and

$S(t)x=\lim_{\lambda\to\infty} e^{-\lambda t} \sum_{n=0}^{\infty} (t^n\lambda^{2n}/n!)(\lambda-Z)^{-n}Cx$ for $x\in X$ and $t\geqq 0$.

Since $\overline{G}\subset Z$, we see that $(\lambda-\overline{G})^{-n}Cx=(\lambda-Z)^{-n}Cx$ for $x\in X$, $n\geqq 0$ and $\lambda>a$. So that $S(t)=S(t)'$ and hence $Z=C^{-1}\overline{G}C$. Q.E.D.

Remark. The following is also proved: A linear operator Z is the generator of an exponentially bounded C-semigroup $\{S(t): t\geqq 0\}$ with $\|S(t)\|\leqq Me^{at}$ if and only if Z is a densely defined closed operator satisfying (a_1)–(a_4) in Theorem 1 and $\{x\in X: Cx\in D(Z)$ and $ZCx\in R(C)\}\subset D(Z)$.

Added in Proof. 1) Corollary can be directly proved without making use of Theorems 1 and 2. 2) For simplicity, we say that $\overline{G}$ is the C-c.i.g. of $\{S(t): t\geqq 0\}$. Recently, Mr. Tanaka has given a characterization for the C-c.i.g. of an exponentially bounded C-semigroup (to appear in Tokyo J. Math.). By using this result we can generate semigroups of the basic classes, such as $(1, A)$, $(0, A)$, $(C_{(k)})$ and growth order α, in a unified way. The details will be published elsewhere.

References

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[3] K. Yosida: Functional Analysis. 6th ed., Springer-Verlag (1980).