65. Characterizations of P^3 and Hyperquadrics Q^3 in P^4

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Introduction. A compact complex threefold is called a Moishezon threefold if it has three algebraically independent meromorphic functions on it. The main consequences we report are

(0.1) **Theorem.** *Let X be a compact complex threefold or a complete irreducible nonsingular algebraic threefold defined over an algebraically closed field of arbitrary characteristic. Assume* $\mathrm{Pic}\,X=ZL$, $H^1(X, \mathcal{O}_X)=0$, $K_X=-dL$, $d\geqq 4$ *(resp.* $d=3$*),* $L^3>0$, *and that* $h^0(X, mL)\geqq 2$ *for some positive integer* m. *Then* X *is isomorphic to projective space* $\boldsymbol{P}^3$ *(resp. a nonsingular hyperquadric* $\boldsymbol{Q}^3$ *in* $\boldsymbol{P}^4$*).*

(0.2) **Theorem.** *A compact complex threefold homeomorphic to* $\boldsymbol{P}^3$ *(resp.* $\boldsymbol{Q}^3$*) is isomorphic to* $\boldsymbol{P}^3$ *(resp.* $\boldsymbol{Q}^3$*) if* $H^q(X, \mathcal{O}_X)=0$ *for any* $q>0$ *and if* $h^0(X, -mK_X)\geqq 2$ *for some positive integer* m.

(0.3) **Theorem.** *A Moishezon threefold homeomorphic to* $\boldsymbol{P}^3$ *(resp.* $\boldsymbol{Q}^3$*) is isomorphic to* $\boldsymbol{P}^3$ *(resp.* $\boldsymbol{Q}^3$*) if its Kodaira dimension is less than three.*

(0.4) **Theorem.** *An arbitrary complex analytic (global) deformation of* $\boldsymbol{P}^3$ *(resp.* $\boldsymbol{Q}^3$*) is isomorphic to* $\boldsymbol{P}^3$ *(resp.* $\boldsymbol{Q}^3$*).*

The theorems (0.2)–(0.4) are derived from (0.1), see section 3. The theorems (0.2)–(0.4) in arbitrary dimension have been proved by Hirzebruch-Kodaira [3] and Yau [14] (resp. by Brieskorn [1]) under the assumption that the manifold is Kählerian. See [2], [5], [6] for related results. Recently Tsuji [12] claims that he is able to prove the theorem (0.4) for $\boldsymbol{P}^n$, whereas Peternell [9] asserts the theorems (0.3) and (0.4) in a stronger form. However there is a gap in the proof of [9], as Peternell himself admits at the end of the article. After the author completed [7] and the major parts of [8], he received two preprints of Peternell [10], [11] in which Peternell completes the proof in [9] of the theorems (0.3) and (0.4) assuming no conditions on Kodaira dimension.

In [7], [8], we make an approach different from theirs and give an elementary proof of the above theorems. Our idea of the proof of (0.1) is as follows. First we see $h^0(X, L)>2$ and then take two distinct members D, D' of the linear system $|L|$. We determine all the possible structures of the scheme-theoretic complete intersection $l=D\cap D'$. From this we easily see that $L^3=1$ (resp. 2), $h^0(X, L)=4$ (resp. 5), and that $|L|$ is base point free. Moreover we see that the morphism associated with $|L|$ is an isomorphism of X onto $\boldsymbol{P}^3$ (resp. $\boldsymbol{Q}^3$).

§1. Proof of (0.1)—the case of projective space $\boldsymbol{P}^3$. In this section we consider the case $d \geqq 4$. We can prove the following lemmas.

(1.1) **Lemma.** *$H^0(X, -mL)=0$ for any $m>0$, $H^3(X, \mathcal{O}_X)=0$.*

(1.2) **Lemma.** *$\chi(X, mL) \geqq (m+1)(m+2)(m+3)/6$.*

(1.3) **Lemma.** *Let D be a reduced and connected effective divisor on X. Then $H^1(X, \mathcal{O}_X(-D))=0$.*

(1.4) **Lemma.** *$h^0(X, L) \geqq 4$.*

(1.5) **Lemma.** *Let D and D' be distinct members of $|L|$, $l:=D \cap D'$ the scheme-theoretic intersection of D and D'. Then $0 \to \mathcal{O}_D(-L) \to \mathcal{O}_D \to \mathcal{O}_l \to 0$ is exact.*

(1.6) **Lemma.** *Let D, D' and $l=D \cap D'$ be the same as in* (1.5). *Then we have,*

(1.6.1) $H^q(X, -mL)=0$
 for $q=0, 1, m>0$; $q=2, 0 \leqq m \leqq d$; $q=3, 0 \leqq m \leqq d-1$,

(1.6.2) $H^q(D, -mL_D)=0$
 for $q=0, m>0$; $q=1, 0 \leqq m \leqq d-1$; $q=2, 0 \leqq m \leqq d-2$,

(1.6.3) $H^0(l, -mL_l)=0$ for $1 \leqq m \leqq d-2$,
 $H^1(l, -mL_l)=0$ for $0 \leqq m \leqq d-3$,

(1.6.4) $H^0(X, \mathcal{O}_X)=H^0(D, \mathcal{O}_D)=H^0(l, \mathcal{O}_l)=\boldsymbol{C}$,

(1.6.5) $H^3(X, -dL)=H^2(D, -(d-1)L_D)=H^1(l, -(d-2)L_l)=\boldsymbol{C}$.

Proof. Clearly any member of $|L|$ is reduced and irreducible. Since $h^0(X, L) \geqq 4$ by (1.4), we can choose distinct D_i's $(i=1, \cdots, m)$ from $|L|$. Hence $D_1+\cdots+D_m$ is reduced and connected. Hence by (1.3), $H^1(X, -mL)=0$ for any $m>0$. Hence $h^2(X, -mL)=h^1(X, -(d-m)L)=0$ for $0 \leqq m \leqq d-1$. It follows that $h^2(X, -dL)=h^1(X, \mathcal{O}_X)=0$. The rest of (1.6.1) is clear from (1.1). The remaining assertions are proved similarly. q.e.d.

(1.7) **Corollary.** $\mathrm{Bs}|L|=\mathrm{Bs}|L_D|=\mathrm{Bs}|L_l|$.

Let D and D' be distinct members of $|L|$, $l:=D \cap D'$ the scheme-theoretic intersection of D and D'. Then l is a pure one dimensional connected closed analytic subspace of X containing $\mathrm{Bs}|L|$, the base locus of the linear system $|L|$. The reduced curve l_{red} consists of nonsingular rational curves intersecting transversally by $H^1(l, \mathcal{O}_l)=0$. It is not difficult to see by using (1.6.3) that $d=4$, $K_X=-4L$ and that there is a unique irreducible component C of l_{red} with $LC=1$.

(1.8) **Lemma.** *Let I_l (resp. I_C) be the ideal sheaf in $\mathcal{O}_X$ defining l (resp. C). Then I_l is not contained in I_C^2.*

(1.9) **Lemma.** *$I_C/I_C^2 \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ and the natural homomorphism $\phi: (I_l/I_l^2) \otimes \mathcal{O}_C \to I_C/I_C^2$ is an isomorphism.*

By (1.9), $I_{l,p}+I_{C,p}^2=I_{C,p}$ whence $I_{l,p}=I_{C,p}$ for any point p of C. This shows that l is nonsingular anywhere along C. Since l is connected by (1.6.4), l is isomorphic to C. Then it is easy to see that $L^3=Ll=1$, $h^0(X, L)=h^0(l, L_l)+2=h^0(l, \mathcal{O}_l(1))+2=4$, and $\mathrm{Bs}|L|=\phi$ by (1.7) and that the morphism associated with $|L|$ is an isomorphism of X onto $\boldsymbol{P}^3$.

§2. Proof of (0.1)—the case of hyperquadrics Q^3 in P^4. In this section, we consider the case $d=3$. Then $h^0(X, L) \geqq 5$. Let D and D' be an arbitrary pair of distinct members of $|L|$, l the scheme-theoretic complete intersection $D\cap D'$ of D and D'. As in section one, we see $H^0(l,\mathcal{O}_l)=C$, $H^1(l,\mathcal{O}_l)=0$ and that l is a pure one dimensional connected closed analytic subspace of X containing $\mathrm{Bs}|L|$.

(2.1) **Lemma.** *l_{red} is a connected (possibly reducible) curve whose irreducible components are nonsingular rational curves intersecting transversally and either*

(2.1.1) *l is a nonsingular rational curve such that $Ll=2$, $I_l/I_l^2\cong\mathcal{O}_l(-2)\oplus\mathcal{O}_l(-2)$, or*

(2.1.2) *l is "a double line" with $l_{\mathrm{red}}(=:C)$ irreducible $LC=1$ such that the ideal I_l (resp. I_C) is given by*

$$I_{l,p}=\mathcal{O}_{X,p}x+\mathcal{O}_{X,p}y^2,\qquad I_{C,p}=\mathcal{O}_{X,p}x+\mathcal{O}_{X,p}y$$

for suitable local parameters x and y at any point p of C,

$$I_C/I_C^2\cong\mathcal{O}_C\oplus\mathcal{O}_C(-1),\quad I_C/I_l\cong\mathcal{O}_C(-1),\quad I_l/I_C^2\cong\mathcal{O}_C,$$

or

(2.1.3) *l_{red} is the union of nonsingular rational curves C and C' with $LC=1$, $LC'=0$, intersecting transversally at a unique point p and except at p, l is a double line along C in the sense of* (2.1.2), *and reduced along C',*

$$I_C/I_C^2\cong\mathcal{O}_C\oplus\mathcal{O}_C(-1),\qquad I_{C'}/I_{C'}^2\cong\mathcal{O}_{C'}(2)\oplus\mathcal{O}_{C'},$$

or

(2.1.4) *l is reduced everywhere and is the union of two rational curves ("lines") C_0, C_m and a (possibly empty) chain of rational curves C_j $(1\leqq j\leqq m-1)$ connecting the "lines" such that $LC_0=LC_m=1$, $LC_j=0$ $(1\leqq j\leqq m-1)$,*

$$I_C/I_C^2=\begin{cases}\mathcal{O}_C\oplus\mathcal{O}_C(-1) & (C=C_0,\ C_m)\\ \mathcal{O}_C(1)\oplus\mathcal{O}_C(1)\quad or\quad \mathcal{O}_C(2)\oplus\mathcal{O}_C & (C=C_1,\cdots,C_{m-1}).\end{cases}$$

It turns out after completing the proof of (0.1) that the case (2.1.3) is impossible and the chain in (2.1.4) is empty.

We infer from (2.1), the following

(2.2) **Lemma.** *$L^3=2$, $h^0(X,L)=5$, the linear system $|L|$ is base point free.*

Then it is easy to prove that X is isomorphic to $\boldsymbol{Q}^3$, noting that any singular hyperquadric has a Weil divisor which is not an integral multiple of a hyperplane section.

§3. Proofs of (0.2) and (0.3).

(3.1) *Proof of* (0.2). By the assumption, $\chi(X,\mathcal{O}_X)=1$ and $H^1(X,\mathcal{O}_X^*)$ $(=:\mathrm{Pic}\,X)\cong H^2(X,Z)\cong H^2(\boldsymbol{P}^3,Z)$ (or $H^2(\boldsymbol{Q}^3,Z))\cong Z$. Let L be a generator of $\mathrm{Pic}\,X$ with $L^3=1$ (resp. $L^3=2$). Then by [3, pp. 207–208] or [1], [6, pp. 317–318], $K_X=-4L$ (resp. $-3L$). Therefore by (0.1), X is isomorphic to $\boldsymbol{P}^3$ (resp. $\boldsymbol{Q}^3$). q.e.d.

(3.2) *Proof of* (0.3). Since X is Moishezon and the Kodaira dimension is less than three, we have $\kappa(X,L)=3$, whence there is a positive integer m such that $h^0(X,mL)\geqq 2$. Refer [4] for $\kappa(X,L)$. By [13, p. 99], $H^q(X,\mathcal{O}_X)$

$=0$ by $b_1=b_3=0$, $b_2=h^{1,1}=1$. Hence X is isomorphic to $\boldsymbol{P}^3$ (resp. $\boldsymbol{Q}^3$) in view of (0.2). q.e.d.

References

[1] Brieskorn, E.: Ein Satz über die komplexen Quadriken. Math. Ann., **155**, 184–193 (1964).

[2] Fujita, T.: On the structure of polarized varieties with Δ-genera zero. J. Fac. Sci. Univ. of Tokyo, **22**, 103–115 (1975).

[3] Hirzebruch, H. and Kodaira, K.: On the complex projective spaces. J. Math. Pure Appl., **36**, 201–216 (1957).

[4] Iitaka, S.: On D-dimensions of algebraic varieties. J. Math. Soc. Japan, **23**, 356–373 (1971).

[5] Kobayashi, S. and Ochiai, T.: Characterizations of complex projective spaces and hyperquadrics. J. Math. Kyoto Univ., **13**, 31–47 (1973).

[6] Morrow, J.: A survey of some results on complex Kähler manifolds. Global Analysis. Univ. of Tokyo Press, pp. 315–324 (1969).

[7] Nakamura, I.: Moishezon threefolds homeomorphic to $\boldsymbol{P}^3$ (preprint).

[8] ——: Moishezon threefolds homeomorphic to a hyperquadric in $\boldsymbol{P}^4$ (preprint).

[9] Peternell, T.: A rigidity theorem for $\boldsymbol{P}_3(\boldsymbol{C})$. Manuscripta Math., **50**, 397–428 (1985).

[10] ——: On the rigidity problem for the complex projective space (preprint).

[11] ——: Algebraic structures on certain 3-folds. Math. Ann., **274**, 133–156 (1986).

[12] Tsuji, H.: Every deformation of $\boldsymbol{P}^n$ is again $\boldsymbol{P}^n$ (preprint).

[13] Ueno, K.: Classification Theory of Algebraic Varieties and Compact Complex Spaces. Lect. Notes in Math., vol. 439, Springer-Verlag (1975).

[14] Yau, S. T.: On Calabi's conjecture and some new results in algebraic geometry. Proc. Nat. Acad. Sci. U.S.A., **74**, 1798–1799 (1977).