64. *Tori whose Covering Spaces have Convex Distance Functions*

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0. **Introduction.** E. Hopf ([4]) proved that Riemannian tori T^2 without conjugate points are flat. The theorem has no analogue in the G-space theory of Busemann ([1]). Namely, H. Busemann ([1], p. 223) has proved that there are metrizations of the torus without conjugate points for which the universal covering space is not Minkowskian. Recently, N. Innami ([5]) proved that Riemannian tori T^n, $n\geq 2$, are flat if there is a point which cannot be a focal point of any geodesic (as a 1-dimensional submanifold). In the present note we shall show that this has an analogue in G-surfaces. The significance of G-spaces can be seen in [1], Section 15.

Let M be a G-space and let $f: M\to \boldsymbol{R}$ be a function. We say that f is *convex* on M if $f\circ\alpha$ is a one-variable convex function for any geodesic α: $(-\infty, \infty)\to M$.

Theorem. *Let N be a G-space which is homeomorphic to the torus T^2 and let M be its universal covering G-space. If M has a point o such that the distance function from o is convex on M, then M is Minkowskian.*

If a compact Riemannian manifold has a non-focal point, then the manifold has no focal points ([6]). And a simply connected Riemannian manifold has no focal points if and only if all distance functions are convex. However, this is not true in the G-space theory. Therefore, we use convex distance functions instead of non-focality properties. We shall show in Section 1 that M is straight, i.e., all geodesics are minimizing in M, and that the distance function from any point is convex on M. Then, combined with the two results, (33.1) in p. 215 and (25.6) in p. 157, [1], these conclude the theorem.

1. **Proof.** We first prove that o is a pole in M, i.e., all geodesics emanating from o is minimizing. Let $\gamma: [0, \infty)\to M$ be a geodesic with $\gamma(0)=o$. Put $f(t)=d(o, \gamma(t))$ for any $t\in[0, \infty)$. Since f is convex and $f(0)=0$,

$$f(t)\geq f'_+(0)t=t$$

for all $t\geq 0$, where $f'_+(0)$ is the right derivative of f at 0 and, hence, $f'_+(0)=1$. This implies that

$$d(o, \gamma(t))=f(t)=t$$

for all $t\geq 0$, because generally $f(t)\leq t$.

Let D be the group of isometries of M such that $M/D=N$. Then, it follows from Proposition 4.1 in [5] that the displacement functions of all isometries of D assume their minimums at o. We now want to prove that the displacement functions of all $\varphi \in D$ are constant on M. Let $1 \neq \varphi \in D$ and let $d_\varphi : M \to \boldsymbol{R}$ given by $d_\varphi(q) = d(q, \varphi q)$ for any $q \in M$ be the displacement function of φ. Suppose there exists a point $p \in M$ such that $d_\varphi(p) > d_\varphi(o) = \min d_\varphi =: L$. Let $\gamma : (-\infty, \infty) \to M$ be the axis of φ through $o = \gamma(0)$, i.e., γ is minimizing and $\varphi\gamma(t) = \gamma(t+L)$ (see [2], p. 66). Choose an isometry $\psi \in D$ such that the point p lies in the strip S bounded by $\gamma(-\infty, \infty)$ and $\psi\gamma(-\infty, \infty)$. Since D is abelian, $\psi\gamma$ is also an axis of φ. Then, as in the proof of Lemma 5.1 in [7], there are a geodesic $\alpha : (-\infty, \infty) \to M$ and a positive $b > L$ such that $\varphi\alpha(t) = \alpha(t+b)$ for any $t \in (-\infty, \infty)$ and $\alpha(-\infty, \infty) \subset S$ (in particular, $d(\alpha(t), \gamma(t))$ is bounded in $t \in (-\infty, \infty)$). Since the Busemann function $f_\gamma(\cdot) := \lim_{t\to\infty}\{d(\cdot, \gamma(t)) - t\} = \lim_{n\to\infty}\{d(\cdot, \gamma(nL)) - nL\}$ is convex, it follows from Corollary 2.3 in [5] that if α is not a co-ray to γ, then $d(\alpha(t), \gamma(t)) \to \infty$ as $t \to \infty$. Thus, α must be an asymptote to γ, and, in particular, is minimizing. This implies that α is also an axis (see [2], p. 65 (2)), namely $b = L$, a contradiction. Therefore, it follows from the proof of Theorem 3.1 in [7] that M is straight.

It remains to prove that the distance function from any point p in M is convex. Let $\alpha : (-\infty, \infty) \to M$ be a geodesic and $f(t) := d(p, \alpha(t))$ for any $t \in (-\infty, \infty)$. We have to prove that

$$f((t_1+t_2)/2) \leq \{f(t_1) + f(t_2)\}/2$$

for any $t_1 \neq t_2 \in (-\infty, \infty)$. To do this we should notice the following. Let $\beta : (-\infty, \infty) \to M$ be a geodesic with $\beta(0) = p$. Then, there exists a sequence of poles o_n $(= \varphi_n o, \varphi_n \in D)$ such that the sequence of geodesics $\beta_n : (-\infty, \infty) \to M$, with $\beta_n(0) = p$ and $\beta_n(d(p, o_n)) = o_n$, converges to β. This fact comes from the proof of (33.1) in p. 215, [1] (in particular, see (2) and (3) in p. 216). Let $\beta : (-\infty, \infty) \to M$ be the geodesic such that $\beta(s) = \alpha((t_1+t_2)/2)$, $s < 0$, and $\beta(0) = p$. Choose a sequence of poles o_n and geodesics β_n as above. Let q_n be the intersection point of $\alpha(t_1, t_2)$ and $\beta_n(-\infty, \infty)$ for sufficiently large n, say $\alpha(\theta_n t_1 + (1-\theta_n)t_2)$, $0 < \theta_n < 1$. Then, by convexity of the distance functions from o_n, we have

$$d(q_n, o_n) \leq \theta_n d(o_n, \alpha(t_1)) + (1-\theta_n) d(o_n, \alpha(t_2))$$

for all n. Since

$$\begin{aligned} d(q_n, o_n) &= d(q_n, p) + d(p, o_n) \\ d(o_n, \alpha(t_1)) &\leq d(o_n, p) + d(p, \alpha(t_1)) \\ d(o_n, \alpha(t_2)) &\leq d(o_n, p) + d(p, \alpha(t_2)), \end{aligned}$$

we have

$$d(q_n, p) \leq \theta_n d(p, \alpha(t_1)) + (1-\theta_n) d(p, \alpha(t_2))$$

for all n. Hence, as $n \to \infty$,

$$f((t_1+t_2)/2) \leq \{f(t_1) + f(t_2)\}/2,$$

because $q_n \to \alpha((t_1+t_2)/2)$ and $\theta_n \to 1/2$. This argument is due to Busemann-Phadke ([3]). This completes the proof of Theorem.

References

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