61. *Class Number Relations of Algebraic Tori.* I

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Let k be an algebraic number field of finite degree and $\mathfrak{p}$ be a place of k. We denote by $k_{\mathfrak{p}}$ the completion of k at the place $\mathfrak{p}$. $O_{\mathfrak{p}}$ denotes the ring of $\mathfrak{p}$-adic integers when $\mathfrak{p}$ is non-archimedean, and $k_{\mathfrak{p}}$ when $\mathfrak{p}$ is archimedean. Thus $U_k=\prod_{\mathfrak{p}} O_{\mathfrak{p}}^{\times}$ is a subgroup of the idele group $k_A^{\times}$. Let T be a torus defined over k and $\hat{T}=\mathrm{Hom}\,(T, G_m)$ be the character module of T. We denote by $T(k)$ the group of k-rational points of T, and by $T(k_{\mathfrak{p}})$ the group of $k_{\mathfrak{p}}$-rational points of T. $T(O_{\mathfrak{p}})$ denotes the unique maximal compact subgroup of $T(k_{\mathfrak{p}})$ when $\mathfrak{p}$ is non-archimedean, and $T(k_{\mathfrak{p}})$ when $\mathfrak{p}$ is archimedean. We put $T(U_k)=\prod_{\mathfrak{p}} T(O_{\mathfrak{p}})$, $T(O_k)=T(U_k)\cap T(k)$ and denote the adele group of T over k by $T(k_A)$. Then $T(U_k)$ is a subgroup of $T(k_A)$. The class number of T over k is defined by

$$h(T)=[T(k_A) : T(k)\cdot T(U_k)].$$

Consider the exact sequence of algebraic tori defined over k

$$(1) \qquad 0\longrightarrow T' \xrightarrow{\alpha} T \xrightarrow{\mu} T'' \longrightarrow 0,$$

where α and μ are defined over k.

Recently, T. Ono treated the case when $T=R_{K/k}(G_m)$ and $T''=G_m$ in (1), where K is a finite Galois extension of k and $R_{K/k}$ is the Weil map. In his paper [3], he defined the number $E(K/k)$ by $h(R_{K/k}(G_m))/h(T')\cdot h(G_m)$ and obtained an equality between $E(K/k)$ and some elementary cohomological invariants of K/k in [4], [5].

In this paper, we shall obtain a similar equality between $h(T)/h(T')\cdot h(T'')$ and some cohomological invariants. Moreover, we shall define a number $E'(K/k)$ for any finite Galois extension K/k and investigate the relation between $E(K/k)$ and $E'(K/k)$.

The author would like to express his hearty thanks to Prof. T. Ono who kindly suggested to him the definition and the importance of the number $E'(K/k)$.

Let A, B be commutative groups and λ be a homomorphism from A to B. If $\mathrm{Ker}\,\lambda$, $\mathrm{Cok}\,\lambda$ are finite, we define the q-symbol of λ by $q(\lambda)=[\mathrm{Cok}\,\lambda]/[\mathrm{Ker}\,\lambda]$. Let $\lambda : T\to T'$ be a k-isogeny of algebraic tori. Then λ induces the following natural homomorphisms

$$\hat{\lambda}(k) : \hat{T}'(k)\longrightarrow \hat{T}(k),$$
$$\lambda(O_{\mathfrak{p}}) : T(O_{\mathfrak{p}})\longrightarrow T'(O_{\mathfrak{p}}),$$
$$\lambda(O_k) : T(O_k)\longrightarrow T'(O_k).$$

Here $\hat{T}(k)$ denotes the submodule of $\hat{T}$ consisting of rational characters defined over k. Then one knows

Lemma (Shyr [6], Theorem 2).

$$\frac{h(T')}{h(T)}=\frac{\tau(T')\,q(\lambda(O_k))\,q(\hat{\lambda}(k))}{\tau(T)\prod_{\mathfrak{p}} q(\lambda(O_{\mathfrak{p}}))},$$

where $\tau(T)$, $\tau(T')$ are the Tamagawa numbers of T, T'.

Corollary. *For any k-isogeny $\gamma: T \longrightarrow T$, we have*

$$1=\frac{q(\gamma(O_k))q(\hat{\gamma}(k))}{\prod_{\mathfrak{p}} q(\gamma(O_{\mathfrak{p}}))}.$$

For any T, T', T'' in (1), one can take a homomorphism $\beta: T\longrightarrow T'$ defined over k, such that $\lambda=\beta\times\mu: T\longrightarrow T'\times T''$ and $\gamma=\beta\cdot\alpha: T'\longrightarrow T'$ are k-isogenies. From Lemma, we have

$$\frac{h(T)\tau(T')\tau(T'')}{\tau(T)h(T')h(T'')}=\frac{\prod_{\mathfrak{p}} q(\lambda(O_{\mathfrak{p}}))}{q(\hat{\lambda}(k))q(\lambda(O_k))}.$$

Let K be the splitting field of T, T', T'' and G be $\mathrm{Gal}(K/k)$. Then, from the exact sequence (1), we have an exact sequence of G-modules

$$(2)\qquad 0\longrightarrow T'(O_K)\longrightarrow T(O_K)\longrightarrow T''(O_K)\longrightarrow 0.$$

Hence we have the following exact sequence derived from (2)

$$(3)\qquad 0\longrightarrow T'(O_k)\xrightarrow{\alpha(O_k)} T(O_k)\xrightarrow{\mu(O_k)} T''(O_k) \longrightarrow H^1(G, T'(O_K))\longrightarrow H^1(G, T(O_K))\longrightarrow\cdots.$$

From this exact sequence, we have

$$[\mathrm{Cok}\,\lambda(O_k)]=[\mathrm{Ker}\,(H^1(G, T'(O_K))\longrightarrow H^1(G, T(O_K)))][\mathrm{Cok}\,\gamma(O_k)],$$

$$[\mathrm{Ker}\,\lambda(O_k)]=[\mathrm{Ker}\,\gamma(O_k)].$$

Therefore we have

$$q(\lambda(O_k))=q(\gamma(O_k))[\mathrm{Ker}\,(H^1(G, T'(O_K))\longrightarrow H^1(G, T(O_K)))].$$

In the same way as above, we obtain $q(\lambda(O_{\mathfrak{p}}))=q(\gamma(O_{\mathfrak{p}}))$ $[\mathrm{Ker}\,(H^1(G_{\mathfrak{P}}, T'(O_{\mathfrak{P}}))\rightarrow H^1(G_{\mathfrak{P}}, T(O_{\mathfrak{P}})))]$ (for all $\mathfrak{p}$), where $\mathfrak{P}$ is an extension of $\mathfrak{p}$ to K and $G_{\mathfrak{P}}$ is the decomposition group of $\mathfrak{P}$. Hence

$$\frac{h(T)\tau(T')\tau(T'')}{\tau(T)h(T')h(T'')}=\frac{\prod_{\mathfrak{p}} q(\gamma(O_{\mathfrak{p}}))[\mathrm{Ker}\,(H^1(G_{\mathfrak{P}}, T'(O_{\mathfrak{P}}))\longrightarrow H^1(G_{\mathfrak{P}}, T(O_{\mathfrak{P}})))]}{q(\hat{\lambda}(k))q(\gamma(O_k))[\mathrm{Ker}\,(H^1(G, T'(O_K))\longrightarrow H^1(G, T(O_K)))]}.$$

By virtue of Corollary, we have

Theorem. *With the notations and assumptions as above, we have*

$$\frac{h(T)\tau(T')\tau(T'')}{\tau(T)h(T')h(T'')}=\frac{q(\hat{\gamma}(k))\prod_{\mathfrak{p}}[\mathrm{Ker}\,(H^1(G_{\mathfrak{P}}, T'(O_{\mathfrak{P}}))\longrightarrow H^1(G_{\mathfrak{P}}, T(O_{\mathfrak{P}})))]}{q(\hat{\lambda}(k))[\mathrm{Ker}\,(H^1(G, T'(O_K))\longrightarrow H^1(G, T(O_K)))]}.$$

Remark. We can regard the formula of T. Ono [4] as a special case of this theorem when $T=R_{K/k}(G_m)$, $T''=G_m$ and $T'=R^{(1)}_{K/k}(G_m)$.

Let K be a finite Galois extension of k again. There exists an exact sequence of algebraic tori defined over k

$$(4)\qquad 0\longrightarrow G_m\longrightarrow R_{K/k}(G_m)\longrightarrow R_{K/k}(G_m)/G_m\longrightarrow 0.$$

We denote by $h'_{K/k}$ the class number of the torus $R_{K/k}(G_m)/G_m$. We shall define the positive rational number $E'(K/k)$ by

$$E'(K/k)=\frac{h_K}{h_k h'_{K/k}},$$

where h_K and h_k are the class numbers of the fields K and k, respectively. By virtue of the fact that $h(R_{K/k}(G_m))=h_K$ and $h(G_m)=h_k$ and our Theorem, we have

$$E'(K/k)=\frac{\prod_{\mathfrak{p}}[H^1(G_{\mathfrak{P}}, O_{\mathfrak{P}}^\times)]}{[H^1(G, O_K^\times)]}=\frac{[H^1(G, U_K)]}{[H^1(G, O_K^\times)]}.$$

It seems to be an interesting problem to investigate the relation of two rational numbers $E(K/k)$ and $E'(K/k)$. When K/k is cyclic, it is easy to show $E(K/k)=E'(K/k)$. One can verify it by calculating the Herbrand quotients of $O_K^\times$ and U_K. But $E'(K/k)$ is not always equal to $E(K/k)$. For example, consider the case when k is an imaginary quadratic field with the class number greater than one, and K is its Hilbert class field. Then we have

$$\frac{E'(K/k)}{E(K/k)}=\frac{[O_k^\times : N_{K/k}O_K^\times]}{[H^3(G, Z)]}\leqq\frac{2}{[H^3(G, Z)]}.$$

From Lyndon's formula, $[H^3(G, Z)]>2$ for any abelian group having non-cyclic p-Sylow subgroup $(p\neq 2)$. As is well known, there exist infinitely quadratic fields which have the ideal class groups of p-rank greater than 2. Therefore $E'(K/k)<E(K/k)$ in these cases.

References

[1] T. Ono: Arithmetic of algebraic tori. Ann. of Math., **74**, 101–139 (1961).
[2] ——: On the Tamagawa number of algebraic tori. ibid., **78**, 47–73 (1963).
[3] ——: A generalization of Gauss' theorem on the genera of quadratic forms. Proc. Japan Acad., **61**A, 109–111 (1985).
[4] ——: On some class number relations for Galois extensions. ibid., **61**A, 311–312 (1985).
[5] ——: On some class number relations for Galois extensions (preprint).
[6] Jih-Min Shyr: On some class number relations of algebraic tori. Michigan Math. J., **24**, 365–377 (1977).