60. *Galois Type Correspondence for Non-separable Normal Extensions of Fields*

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In this paper we try to extend the classical Galois-Krull theory for separable and normal extensions of fields, and the Jacobson theory for finite purely inseparable extensions of exponent 1, to general normal extensions of exponent 1 (i.e., to those extensions whose maximal pure subextensions have exponent 1).

A. **Definition 1.** An algebraic extension of fields K/k will be called *distinguished* if it is possible to find a purely inseparable subextension $L/k \subset K/k$ with K/L separable.

Proposition 1. *Let K/k be a distinguished extension of fields, k_K/k the maximal separable subextension of K/k, and K_0/k the maximal pure subextension of K/k. In this case $K=K_0\cdot k_K$, and K/K_0 is separable. If N/k is pure and M/k is separable, the compositum $N\cdot M/k$ is distinguished.*

Corollary 1. *A separable, purely inseparable, or normal extension K/k is distinguished.*

Proposition 2. *Every algebraic extension K/k contains a maximal distinguished subextension K_d/k.*

B. Let K/k be a normal extension of fields of characteristic $p\neq 0$, of exponent 1 (i.e., such that K_0/k has exponent 1). In the following we conserve the notations from Proposition 1. We denote by $\mathcal{D}_{K/k}$, the K-linear space of all k-derivations of K, and by S the group $\mathrm{Aut}(K/k)$. It is clear that $K_0=K^S=\{x\in K, \sigma(x)=x, \text{ for every } \sigma\in S\}$. For a K-subspace $\mathcal{A}$ of $\mathcal{D}_{K/k}$, denote by $N(\mathcal{A})$ and the annulator $\bigcap_{D\in\mathcal{A}}\mathrm{Ker}\,D$ of $\mathcal{A}$, and for a subextension $L/k\subset K/k$ denote by $\mathcal{A}(L)$ the K-subspace $\{D\in\mathcal{D}_{K/k}, D(x)=0 \text{ for all } x\in L\}$ of $\mathcal{D}_{K/k}$.

Definition 2. A K-subspace $\mathcal{A}$ of $\mathcal{D}_{K/k}$, will be called *arithmetically maximal* (*A-maximal*) if for any other K-subspace $\mathcal{B}$ of $\mathcal{D}_{K/k}$ with $N(\mathcal{B})=N(\mathcal{A})$ and $\mathcal{B}\supset\mathcal{A}$, we have $\mathcal{B}=\mathcal{A}$.

Corollary 2. *$\mathcal{A}$ is an A-maximal K-subspace of $\mathcal{D}_{K/k}$ if and only if $\mathcal{A}(N(\mathcal{A}))=\mathcal{A}$.*

For a derivation $D\in\mathcal{D}_{K_0/k}$ we denote by D^* the unique derivation in $\mathcal{D}_{K/k}$ which extends D([3], Chapter. X, Theorem 7 and conseq.). Note that the application $D\to D^*$ is K_0-linear and we can view $\mathcal{D}_{K_0/k}$ as a K_0-subspace in $\mathcal{D}_{K/k}$.

Definition 3. The set $G(K/k)=S\times\mathcal{D}_{K/k}$ becomes a group with the natural componentwise group operation ($\mathcal{D}_{K/k}$ is considered as an additive group). It is called *the Galois group of rank* 2 *associated with* K/k. We put now $G_0(K/k)=S\times\mathcal{D}_{K_0/k}$ and call it *the dual Galois group of rank* 2 *associated with* K/k. It is clear that $G_0(K/k)$ is a subgroup in $G(K/k)$.

Lemma 1. *For any* $\sigma\in S$ *and* $D\in\mathcal{D}_{K_0/k}$, *we have* $\sigma D^*=D^*\sigma$.

Definition 4. A subgroup $M=(H,\mathcal{A})$ in $G_0(K/k)$ is said to be *closed*, if H is closed in the Krull topology on $S=\mathrm{Aut}(k_K/k)$, and $\mathcal{A}$ is an A-maximal K-subspace of $\mathcal{D}_{K_0/k}$. Note that $S=\mathrm{Aut}(K/k)$.

For a subextension $L/k\subset K/k$, put $\mathcal{A}_0(L)=\{D\in\mathcal{D}_{K_0/k},\ D^*(x)=0$ for all $x\in L\}$, $\psi(L)=M_L=(H_L,\mathcal{A}_L)$, where $H_L=\{\sigma\in S,\ \sigma(x)=x$ for all $x\in L\}$, $\mathcal{A}_L=\mathcal{A}_0(L\cap K_0)$, and $\varphi(M)=L_M=(\mathrm{Fix}\,H\cap k_K)N_0(\mathcal{A})$ for $M=(H,\mathcal{A})\subset G_0(K/k)$, and $N_0(\mathcal{A})=\{x\in K_0,\ D(x)=0$ for all $D\in\mathcal{A}\subset\mathcal{D}_{K_0/k}\}$.

Theorem 1. *Let* K/k *be a normal algebraic extension of exponent* 1. *With the above notations, the maps* ψ *and* φ *establish a one-to-one correspondence between the distinguished subextensions of* K/k *and the closed subgroups of* $G_0(K/k)$.

Definition 5. A subgroup $M=(H,\mathcal{A})$ is called *admissible* if H is closed in the Krull topology on S, $\mathcal{A}$ is an A-maximal K-subspace in $\mathcal{D}_{K/k}$, and if we can find a p-base $\{c_i\}$ of $N(\mathcal{A})$ over k_K such that $c_i\in\mathrm{Fix}\,H$, for all i.

Theorem 2. *Let* K/k *be a normal extension of exponent* 1. *The maps* $\bar{\psi}(L)=(H_L,\mathcal{A}_L)\subset G(K/k)$ *with* $H_L=\{\sigma\in S,\ \sigma(x)=x$ *for all* $x\in L\}$, $\mathcal{A}_L=\{D\in\mathcal{D}_{K/k},\ D(x)=0$ *for* $x\in L\}$, *and* $\bar{\varphi}(H,\mathcal{A})=\mathrm{Fix}\,H\cap N(\mathcal{A})$, *establish a one-to-one correspondence between the arbitrary subextensions* $L/k\subset Kk$ *and the admissible subgroups* $(H,\mathcal{A})$ *in* $G(K/k)$.

Remark. For K/k purely inseparable, finite and of exponent 1, the A-maximal K-subspaces in $\mathcal{D}_{K/k}$ are exactly the *restricted Lie algebras* of Jacobson [1]. When K/k is infinite, purely inseparable, and of exponent 1, the A-maximal K-subspaces in $\mathcal{D}_{K/k}$ are exactly the closed (in the finite topology on $\mathcal{D}_{K/k}$) K-subspaces are closed for taking p-powers ([4], [5], [6]). We have proved the same result, independently and with other tools. It is not difficult to prove that when K/k is normal with K_0/k of exponent 1, the A-maximal subspaces are exactly the K-subspaces, closed in the finite topology on $\mathcal{D}_{K/k}$, and closed for taking p-powers.

The point in order to establish this assertion is the following result:

Lemma 2. *Let* K/k *be a purely inseparable* (*finite or not*) *extension of exponent* 1, *of characteristic* p, *and* $K\supset L\supset k$. *Then there exists a derivation* D *of* K/k *such that* $\mathrm{Ker}\,D=L$.

Proof. Following an idea of Gerstenhaber [5] we define, for a fixed p-base B of K/L, $D(c_i)=c_i^{p+1}$, where c_i runs in B and D is 0 on L. Let now C be a linear combination over L consisting of monomials of the form $M=c_1^{i_1}\cdots c_t^{i_t}$, where $c_i\in B$, and $0\leqq i_1,\cdots,i_t\leqq p-1$. Suppose $D(C)=0$. But we can consider that the monomials M from C are L-independent. In $D(C)$ all the monomials M appear with the coefficient $i_1c_1^p+\cdots+i_tc_t^p$, and we may conclude that $i_1c_1^p + \cdots + i_tc_t^p = 0$ for a monomial $M \neq 1$, if C is not trivial. This is now a contradiction because $c_1^p, \cdots, c_t^p$ are independent over the prime field of L. So we have Ker $D = L$.

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