# 58. *Simple Proof of Fermat's Last Theorem for $n=4$*

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In this paper, we shall give a simple proof of the following theorem which is slightly stronger than, and implies Fermat's last theorem for $n=4$.

**Theorem 1.** *Let $m$ be a non-negative integer. Then the equation*
$$x^4+2^m y^4=z^4$$
*has no solutions in odd integers $x, y, z$.*

**Theorem 2.** *The equation $X^4+Y^4=Z^4$ has no solutions in positive integers.*

*Proof.* Let $d$ be a greatest common divisor of $X$ and $Y$. It would follow that $d$ divided $Z$. Set $X=dx$, $Y=dy$, $Z=dz$. Then $x$ and $y$ are relatively prime. If $x$ and $y$ are both odd integers, since the square of an odd integer is congruent to 1 modulo 4, we have
$$x^4+y^4\equiv 2 \pmod 4). \quad \text{But } z^4\equiv 0 \pmod 4 \quad \text{or} \quad z^4\equiv 1 \pmod 4,$$
and hence $x$ and $y$ must be of opposite parity and without loss of generality we may assume that $y$ is even. Let $y=2^m y_0$ where $m$ is a positive integer and $y_0$ is an odd integer. Therefore we obtain
$$x^4+2^{4m}y_0^4=z^4$$
where $x, y_0, z$ are all odd integers. Thus it is sufficient to complete the proof of this Fermat's last theorem for $n=4$ that we prove Theorem 1.

*Proof of Theorem* 1. Suppose that $u$ is the least number for which $x^4+2^m y^4=u^4$ has a solution in positive odd integers $x, y, u$ for some non-negative integer $m$. The statement that $u$ is least immediately implies that three integers $x, y, u$ are pairwise relatively prime. Since the fourth power of an odd integer is congruent to 1 modulo 16, we have
$$2^m y^4=u^4-x^4\equiv 1-1=0 \pmod{16}.$$
Then $m>3$. Since $u$ and $x$ are both odd and relatively prime, we have
$$u^2+x^2\equiv 2 \pmod 4$$
and
$$(u^2+x^2,\ u+x)=(u^2+x^2,\ u-x)=(u+x,\ u-x)=2.$$
And since
$$2^m y^4=u^4-x^4=(u-x)(u+x)(u^2+x^2),$$
there are positive odd integers $A, B, C$ such that
$$u-x=2A, \quad u+x=2^{m-2}B, \quad u^2+x^2=2C$$
or
$$u-x=2^{m-2}B, \quad u+x=2A, \quad u^2+x^2=2C.$$

Hence $y^4=ABC$ and $A, B, C$ are pairwise relatively prime. Thus all of $A, B, C$ must be fourth powers, say $A=a^4$, $B=b^4$, $C=c^4$. Then

$$4c^4=2(u^2+x^2)=(u-x)^2+(u+x)^2=4a^8+2^{2m-4}b^8$$

and so we obtain

$$(a^2)^4+2^{2m-6}(b^2)^4=c^4$$

in positive odd integers $a$, $b$, $c$.

Moreover, since $0<x<u$, we have $c^4<2c^4=u^2+x^2<2u^2<u^4$ and so $0<c<u$. Thus $u$ was not least after all and the theorem is proved.