51. *Magnetohydrodynamic Approximation of the Complete Equations for an Electromagnetic Fluid. II*

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1. Introduction and equations. In the previous paper [2], we justified the magnetohydrodynamic approximation locally in time for certain two-dimensional flow of an electrically conducting compressible fluid. It was proved that the magnetohydrodynamic equations were obtained as the singular limit of the complete equations at the vanishing of the dielectric constant. The aim of this note is to justify the approximation globally in time in case where the fluid is viscous and heat-conductive.

The equations considered are

$$\begin{aligned} &\rho_t+\mathrm{div}\,(\rho u)=0,\\ &\rho(u_t+(u\cdot\nabla)u)+\nabla p=\mathrm{div}\,(2\mu P+\mu' I\,\mathrm{div}\,u)+J\times B,\\ (1)\qquad &\rho e_\theta(\theta_t+u\cdot\nabla\theta)+\theta p_\theta\,\mathrm{div}\,u=\mathrm{div}\,(\kappa\nabla\theta)+\Psi+J(E+u\times B),\\ &\varepsilon E_t-(1/\mu_0)\,\mathrm{rot}\,B+J=0,\\ &B_t+\mathrm{rot}\,E=0,\\ (2)\qquad &\mathrm{div}\,B=0. \end{aligned}$$

Here and in the sequel, we use the notations for two-dimensional vectors. The unknowns $\rho>0$, $u=(u^1, u^2)$, $\theta>0$, E (scalar) and $B=(B^1, B^2)$ represent the mass density, the velocity, the absolute temperature, the electric field and the magnetic flux density, respectively. They are functions of time $t\geq 0$ and space variable $x=(x_1, x_2)\in \boldsymbol{R}^2$. The pressure p and the internal energy e are smooth functions of (ρ, θ) such that $p_\rho=\partial p/\partial\rho>0$ and $e_\theta=\partial e/\partial\theta>0$. The thermodynamic law $de=\theta dS-pd(1/\rho)$ is always assumed, where S (the entropy) is a smooth function of (ρ, θ). P is the deformation tensor, whose entries are $P_{ij}=(1/2)(\partial_j u^i+\partial_i u^j)$, $i, j=1, 2$, where $\partial_i=\partial/\partial x_i$.

$$\Psi=2\mu\sum P_{ij}^2+\mu'(\mathrm{div}\,u)^2$$

is the viscous dissipation function. The current density J (scalar) is given by Ohm's law $J=\sigma(E+u\times B)$. The viscosity coefficients μ and μ', the heat conductivity coefficient κ and the electric conductivity coefficient σ are smooth functions of (ρ, θ) such that $\mu>0$, $2\mu+\mu'>0$, $\kappa>0$ and $\sigma>0$. The dielectric constant ε and the magnetic permeability μ_0 are assumed to be positive constants.

The magnetohydrodynamic equations corresponding to (1), (2) are given by

$$\begin{aligned} &\rho_t+\mathrm{div}\,(\rho u)=0,\\ (3)\qquad &\rho(u_t+(u\cdot\nabla)u)+\nabla p-(1/\mu_0)\,\mathrm{rot}\,B\times B=\mathrm{div}\,(2\mu P+\mu' I\,\mathrm{div}\,u),\\ &\rho e_\theta(\theta_t+u\cdot\nabla\theta)+\theta p_\theta\,\mathrm{div}\,u=\mathrm{div}\,(\kappa\nabla\theta)+\Psi+(1/\sigma\mu_0^2)(\mathrm{rot}\,B)^2, \end{aligned}$$

$$B_t - \mathrm{rot}\,(u\times B) = -\mathrm{rot}\,\{(1/\sigma\mu_0)\,\mathrm{rot}\,B\},$$

(4) $\quad \mathrm{div}\,B=0.$

In this case, the electric field is determined by the relation

(5) $$E=E(\rho, u, \theta, B)\equiv -u\times B+(1/\sigma\mu_0)\,\mathrm{rot}\,B.$$

2. Global existence. We consider the system (1), (2) with the initial data

(6) $$(\rho, u, \theta, E, B)(0, x)=(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x).$$

Let $s\geq 3$ be an integer, and let $\bar\rho>0$, $\bar\theta>0$ and $\bar B\in \boldsymbol{R}^2$ be fixed constants. We assume the following conditions on the initial data.

(7) For each $\varepsilon\in(0,1]$, $(\rho_0^\varepsilon-\bar\rho, u_0^\varepsilon, \theta_0^\varepsilon-\bar\theta, E_0^\varepsilon, B_0^\varepsilon-\bar B)\in H^s$, $\inf_x\{\rho_0^\varepsilon(x), \theta_0^\varepsilon(x)\}>0$, and $\mathrm{div}\,B_0^\varepsilon=0$ on $\boldsymbol{R}^2$.

(8) $$\sup_\varepsilon \|(\rho_0^\varepsilon-\bar\rho, u_0^\varepsilon, \theta_0^\varepsilon-\bar\theta, \varepsilon^{1/2}E_0^\varepsilon, B_0^\varepsilon-\bar B)\|_s=K_1<+\infty.$$

Let us denote by $X^s(T)$ the set of all functions $(\rho, u, \theta, E, B)(t, x)$ satisfying the following conditions: $(\rho-\bar\rho, u, \theta-\bar\theta, E, B-\bar B)\in C^0(0, T; H^s)$, $D_x(u,\theta)\in L^2(0,T; H^s)$, $\partial_t(\rho, E, B)\in C^0(0,T; H^{s-1})$ and $\partial_t(u,\theta)\in C^0(0,T;H^{s-2})\cap L^2(0,T;H^{s-1})$. Let $\varepsilon\in(0,1]$ and $t\in[0,T]$. For $U=(\rho,u,\theta,E,B)\in X^s(T)$, we define $M_\varepsilon(t;U)$ by

(9) $$M_\varepsilon(t;U)^2=\sup_{0\leq\tau\leq t}\|(\rho-\bar\rho, u, \theta-\bar\theta, \varepsilon^{1/2}E, B-\bar B)(\tau)\|_s^2 + \int_0^t \|D_x(\rho, E, B)(\tau)\|_{s-1}^2+\|D_x(u,\theta)(\tau)\|_s^2+\|(E+u\times\bar B)(\tau)\|_s^2\,d\tau.$$

The following result concerning the existence of global solution of (1), (2) is proved in the same way as in [1] (Theorem 3.2).

Theorem 1. *Assume* (7) *and* (8). *Then there exists a positive constant δ_1 independent of $\varepsilon\in(0,1]$ such that if $K_1\leq\delta_1$, the initial value problem* (1), (2), (6) *has a unique global solution $U^\varepsilon=(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)\in X^s(+\infty)$, which satisfies the estimate $M_\varepsilon(t;U^\varepsilon)\leq C_1K_1$ for any $t\in[0,\infty)$. Here $C_1=C_1(\delta_1)$ is a constant independent of ε. Furthermore, for each ε, the solution converges to the constant state $(\bar\rho, 0, \bar\theta, 0, \bar B)$ uniformly in $x\in\boldsymbol{R}^2$ as $t\to\infty$.*

3. Estimates for time derivatives. In order to get sharp estimates for time derivatives of the solution, we make additional hypotheses for the initial data:

$(10)_1$ $$\sup_\varepsilon \varepsilon^{-\beta}\|E_0^\varepsilon-E(\rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon, B_0^\varepsilon)\|_{s-1}=K_2<+\infty,$$

$(10)_2$ $$\sup_\varepsilon \varepsilon^{(1/2-\beta')}\|\mathrm{rot}\,E_0^\varepsilon\|_{s-1}=K_3<+\infty,$$

where $\beta\geq 0$ and $\beta'\in[0,1/2]$ are independent of $\varepsilon\in(0,1]$. $E(\rho,u,\theta,B)$ is the function in (5).

Let $\varepsilon\in(0,1]$ and $t\in[0,T]$. For $U=(\rho,u,\theta,E,B)\in X^s(T)$ and $\eta\in[0,1]$, we define $N_\varepsilon(t;U,\eta)$ by

(11) $$N_\varepsilon(t;U,\eta)^2=\sup_{0\leq\tau\leq t}\|\partial_t\rho(\tau)\|_{s-1}^2+\int_0^t\|\partial_t(\rho,u,\theta,B)(\tau)\|_{s-1}^2\,d\tau + \varepsilon^\eta\Big\{\sup_{0\leq\tau\leq t}\|\partial_t(\varepsilon^{1/2}E, B)(\tau)\|_{s-1}^2+\int_0^t\|\partial_t E(\tau)\|_{s-1}^2\,d\tau\Big\}.$$

Proposition 2. *Assume* (7), (8) *and* $(10)_{1,2}$. *Then there exists a positive* *constant* δ_2 ($\leq\delta_1$) *independent of* $\varepsilon\in(0,1]$ *such that if* $K\equiv K_1+K_2+K_3\leq\delta_2$, *the solution* $U^\varepsilon=(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)\in X^s(+\infty)$ *constructed in Theorem* 1 *satisfies the estimate* $N_\varepsilon(t; U^\varepsilon, \eta)+\|E^\varepsilon(t)\|_{s-1}\leq C_2K$ *for any* $t\in[0,\infty)$. *Here* $\eta=\max\{1-2\beta, 1-2\beta'\}\in[0,1]$, *and* $C_2=C_2(\delta_2)$ *is a constant independent of* ε.

We give the outline of the proof. Let $R_\varepsilon(t)=\sup\{\|(E^\varepsilon+u^\varepsilon\times\bar{B})(\tau)\|_{s-1}; 0\leq\tau\leq t\}$. By the argument similar to that employed in the proof of Proposition 3.1 of [2], we get the inequalities

$$\|\partial_t\rho^\varepsilon(t)\|_{s-1}^2+\int_0^t\|\partial_t(\rho^\varepsilon, u^\varepsilon, B^\varepsilon)(\tau)\|_{s-1}^2\,d\tau\leq CK_1^2, \tag{12}$$

$$\int_0^t\|\partial_t\theta^\varepsilon(\tau)\|_{s-1}^2\,d\tau\leq CK_1^2+CK_1R_\varepsilon(t)^2, \tag{13}$$

$$\varepsilon^\eta\left\{\|\partial_t(\varepsilon^{1/2}E^\varepsilon, B^\varepsilon)(t)\|_{s-1}^2+\int_0^t\|\partial_tE^\varepsilon(\tau)\|_{s-1}^2\,d\tau\right\}\leq CK^2+C(K_1+R_\varepsilon(t))N_\varepsilon(t; U^\varepsilon, \eta)^2, \tag{14}$$

$$R_\varepsilon(t)\leq C(K_1+N_\varepsilon(t; U^\varepsilon, \eta)). \tag{15}$$

Here we used the estimate $M_\varepsilon(t; U^\varepsilon)\leq C_1K_1$. $C=C(\delta_1)$ is a constant independent of ε. The conclusion of the proposition follows from the combination of the inequalities (12)–(15).

4. Convergence as $\varepsilon\to 0$. In addition to the conditions (7), (8) and $(10)_{1,2}$, we assume the following: There is a function $(\rho_0^0, u_0^0, \theta_0^0, B_0^0)(x)$ with $(\rho_0^0-\bar{\rho}, u_0^0, \theta_0^0-\bar{\theta}, B_0^0-\bar{B})\in H^s$ such that

$$\sup_\varepsilon \varepsilon^{-\gamma}\|(\rho_0^\varepsilon-\rho_0^0, u_0^\varepsilon-u_0^0, \theta_0^\varepsilon-\theta_0^0, B_0^\varepsilon-B_0^0)\|_{s-1}=K_4<+\infty, \tag{16}$$

where $\gamma>0$ is independent of $\varepsilon\in(0,1]$.

Let $Y^s(T)$ be the set of all functions $(\rho, u, \theta, B)(t, x)$ satisfying the following conditions: $(\rho-\bar{\rho}, u, \theta-\bar{\theta}, B-\bar{B})\in C^0(0, T; H^s)$, $D_x(u, \theta, B)\in L^2(0, T; H^s)$, $\partial_t\rho\in C^0(0, T; H^{s-1})$ and $\partial_t(u, \theta, B)\in C^0(0, T; H^{s-2})\cap L^2(0, T; H^{s-1})$.

Theorem 3. *Assume* (7), (8), $(10)_{1,2}$ *and* (16). *Then there exists a positive constant* δ_3 ($\leq\delta_2$) *independent of* $\varepsilon\in(0,1]$ *such that if* $K\equiv K_1+K_2+K_3\leq\delta_3$, *the solution* $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)\in X^s(+\infty)$ *constructed in Theorem* 1 *satisfies the following properties: Let* $T>0$ *be arbitrary. Then* $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon, E^\varepsilon, B^\varepsilon)(t, x)$ *converges on* $[0, T]\times\boldsymbol{R}^2$ *to a function* $(\rho^0, u^0, \theta^0, E^0, B^0)(t, x)$ *as* $\varepsilon\to 0$. *The limit function* $(\rho^0, u^0, \theta^0, B^0)(t, x)$ *excepting* $E^0(t, x)$ *is a unique solution in* $Y^s(T)$ *of the magnetohydrodynamic equations* (3), (4), *with the initial condition* $(\rho^0, u^0, \theta^0, B^0)(0, x)=(\rho_0^0, u_0^0, \theta_0^0, B_0^0)(x)$. *Also, the equation* (5) *holds for the limit function. Moreover, the following estimate holds for* $t\in[0, T]$ *and* $\varepsilon\in(0,1]$.

$$\|(\rho^\varepsilon-\rho^0, u^\varepsilon-u^0, \theta^\varepsilon-\theta^0, B^\varepsilon-B^0)(t)\|_{s-1}^2+\int_0^t\|(u^\varepsilon-u^0, \theta^\varepsilon-\theta^0)(\tau)\|_s^2+\|(E^\varepsilon-E^0)(\tau)\|_{s-1}^2\,d\tau\leq\varepsilon^{2\lambda}C_3K^2e^T, \tag{17}$$

where $C_3=C_3(\delta_3)$ *is a constant independent of* ε, *and* $\lambda=\min\{\gamma, 1-\eta/2\}>0$ (η *is determined in Proposition* 2).

This result can be proved by the energy method employed in the proof of Theorem 5.1 of [2]. We omit the details.

References

[1] S. Kawashima: Smooth global solutions for two-dimensional equations of electro-magneto-fluid dynamics. Japan J. Appl. Math., 1, 207–222 (1984).
[2] S. Kawashima and Y. Shizuta: Magnetohydrodynamic approximation of the complete equations for an electromagnetic fluid (to appear in Tsukuba J. Math.).