50. *Initial-boundary Value Problem for Parabolic Equation in L^1*

By Dong Gun PARK
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1986)

Let Ω be a not necessarily bounded domain in R^n locally regular of class C^{2m} and uniformly regular of class C^m in the sense of F. E. Browder [4]. We consider the following parabolic initial-boundary value problem

(1) $$\partial u/\partial t+A(x,t,D)u=f(x,t),\quad x\in\Omega,\quad 0<t\leqq T,$$

(2) $$B_j(x,t,D)u=0,\quad j=1,\cdots,m/2,\quad x\in\partial\Omega,\quad 0<t\leqq T,$$

(3) $$u(x,0)=u_0(x),\qquad x\in\Omega,$$

in $L^1(\Omega)$. Here for each $t\in[0,T]$

$$A(x,t,D)u=\sum_{|\alpha|\leqq m}a_\alpha(x,t)D^\alpha$$

is a strongly elliptic linear differential operator of order m and

$$B_j(x,t,D)=\sum_{|\beta|\leqq m_j}b_{j\beta}(x,t)D^\beta,\qquad j=1,\cdots,m/2,$$

is a normal set of linear differential operators on $\partial\Omega$ of order less than m. Similar problem was discussed in [3], [9], [10] for equations with coefficients independent of t. In [3] with the aid of the theory of dual semigroups H. Amann showed that the associated elliptic operator generates an analytic semigroup in $L^1(\Omega)$ in case $m=2$.

Concerning the coefficients of $A(x,t,D)$ and $B_j(x,t,D)$ we assume the following regularity conditions:

(i) $a_\alpha(x,t)$, $|\alpha|=m$, and their derivatives $\partial a_\alpha(x,t)/\partial t$ with respect to t are bounded and uniformly continuous in $\bar\Omega\times[0,T]$.

(ii) $a_\alpha(x,t)$, $|\alpha|<m$, and their derivatives with respect to t are bounded and measurable, and continuous in t uniformly in $\bar\Omega\times[0,T]$.

(iii) The coefficients of $B_j(x,t,D)$ are extended to $\bar\Omega\times[0,T]$ so that $(\partial/\partial x)^\gamma b_{j\beta}(x,t)$, $(\partial/\partial t)(\partial/\partial x)^\gamma b_{j\beta}(x,t)$, $|\beta|\leqq m_j$, $|\gamma|\leqq m-m_j$, $j=1,\cdots,m/2$, are bounded and uniformly continuous in $\bar\Omega\times[0,T]$.

(iv) The formally constructed adjoint boundary value problem $(A'(x,t,D)$, $\{B'_j(x,t,D)\}_{j=1}^{m/2},\Omega)$ satisfies (i), (ii), (iii).

For the well-posedness of the problem (1)–(3) we assume that for each fixed $t\in[0,T]$ and $\theta\in[\pi/2,3\pi/2]$

$$(-1)^{m/2}e^{i\theta}(\partial/\partial\tau)^m+A(x,t,D),\qquad\{B_j(x,t,D)\}_{j=1}^{m/2}$$

satisfies the complementing condition in the cylindrical domain $\Omega\times(-\infty,\infty)$ ([1], [2]).

The operator $A(t)$ is defined as follows:

The domain $D(A(t))$ is the totality of functions u satisfying

(i) $u\in W^{m-1,q}(\Omega)$ for each $q\in[1,n/(n-1))$,

(ii) $A(x, t, D)u \in L^1(\Omega)$ in the distribution sense,

(iii) for any p such that $0<(n/m)(1-1/p)<1$ and for any $v \in D(A'_p(t))$

$$(A(x, t, D)u, v)=(u, A'(x, t, D)v),$$

where $A'_p(t)$ is the realization of $A'(x, t, D)$ in $L^p(\Omega)$ under the boundary conditions $B'_j(x, t, D)u|_{\partial\Omega}=0,\ j=1, \cdots, m/2$;

and for $u \in D(A(t))$ $A(t)u=A(x, t, D)u$.

If $m_j<m-1$, the trace of $B_j(x, t, D)u$ on $\partial\Omega$ is defined and vanishes for $u \in D(A(t))$. It is known that $-A(t)$ generates an analytic semigroup in $L^1(\Omega)$ ([9], [10]).

Let $A_p(t)$, $1<p<\infty$, be the realization of $A(x, t, D)$ in $L^p(\Omega)$ under the boundary conditions $B_j(x, t, D)u|_{\partial\Omega}=0,\ j=1, \cdots, m/2$. Assuming in addition that Ω is bounded but without the assumption on the adjoint boundary value problem A. Yagi [11] showed the existence of the evolution operator to the equation

$$du(t)/dt+A_p(t)u(t)=f(t). \tag{4}$$

In this paper we show that for some $\rho \in (0, 1]$ $A(t)^\rho \cdot dA(t)^{-1}/dt$ is uniformly bounded:

$$\|A(t)^\rho \cdot dA(t)^{-1}/dt\| \leqq C, \tag{5}$$

and hence we can apply the result of [7] to construct the evolution operator to the equation in $L^1(\Omega)$:

$$du(t)/dt+A(t)u(t)=f(t). \tag{6}$$

As was remarked in [12] the condition (5) is essentially equivalent to the condition of Yagi [11]:

$$\|A(t)(\lambda-A(t))^{-1} \cdot dA(t)^{-1}/dt\| \leqq N/|\lambda|^\rho.$$

Theorem 1. *Under the hypothesis stated above the evolution operator $U(t, s)$ to the equation* (6):

$$(\partial/\partial t)U(t, s)+A(t)U(t, s)=0, \qquad U(s, s)=I,$$
$$(\partial/\partial s)U(t, s)-U(t, s)A(s)=0 \qquad on\ D(A(s))$$

exists, and satisfies

$$\|(\partial/\partial t)U(t, s)\|=\|A(t)U(t, s)\| \leqq C/(t-s),$$

$A(t)U(t, s)A(s)^{-1}$ is strongly continuous in $0 \leqq s \leqq t \leqq T$.

Outline of proof. We denote by $(\cdot, \cdot)_{\theta, q}$ the real interpolation space. In view of P. Grisvard [6] for any $\theta \in (0, 1)$

$$W^{\theta,1}(\Omega)=(L^1(\Omega), W_0^{m,1}(\Omega))_{\theta/m,1}$$

where $W_0^{m,1}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,1}(\Omega)$. It is easy to show that

$$(L^1(\Omega), D(A(t)))_{\theta/m,1} \subset D(A(t)^\rho)$$

if $0<\rho<\theta/m$. Hence it follows that $W^{\theta,1}(\Omega) \subset D(A(t)^\rho)$ since $W_0^{m,1}(\Omega) \subset D(A(t))$. Thus in order to establish (5) it suffices to show that

$$(d/dt)A(t)^{-1}f \in W^{m-1,1}(\Omega) \tag{7}$$

for any $f \in L^1(\Omega)$ since clearly $W^{m-1,1}(\Omega) \subset W^{\theta,1}(\Omega)$. The relation (7) is established following the method of estimating the kernels of $\exp(-\tau A)$ and $(A-\lambda)^{-1}$ in [8], [10] where $A=A(t)$ for some fixed t.

Theorem 2. *The operator $U(t, s)$ has a kernel $G(x, y, t, s)$ satisfying*

$$|G(x, y, t, s)| \leqq \frac{C}{(t-s)^{n/m}} \exp\left(-c\frac{|x-y|^{m/(m-1)}}{(t-s)^{1/(m-1)}}\right). \tag{8}$$

Outline of proof. Following [8], [10] one can show that

$$R_1(t, s) = -(\partial/\partial t + \partial/\partial s) \exp(-(t-s)A(t))$$

has a kernel which satisfies the same type of estimate as (8). Hence, the result follows by the same argument as that of S. D. Eidel'man [5: pp. 73–75].

The author wishes to express the deepest appreciation to Prof. H. Tanabe, the author's advisor, for his valuable suggestions and encouragement.

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