47. On the Existence and the Asymptotic Behavior of the Global Solution of a Nonlinear Variational Inequality of Evolution

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§1. **Introduction.** In this paper we consider the existence and the asymptotic behavior of the global solution of the following variational inequality of evolution (1.1) associated with the boundary condition (1.2) and the initial condition (1.3):

(1.1) $\quad (u_t-\Delta u-e^u-f,\ v-u)_{L^2(\Omega)}\geqq 0 \qquad$ for a.e. $t\in(0,\infty)$,

for any $v\in L^2(\Omega)$ satisfying $0\leqq v\leqq M$ a.e. in Ω, and $0\leqq u\leqq M$ on $[0,\infty]\times\bar{\Omega}$,

(1.2) $\quad \alpha u+(1-\alpha)(\partial u/\partial \boldsymbol{n})=0 \qquad$ on $(0,\infty)\times\partial\Omega$,

(1.3) $\quad u(0,\cdot)=u_0 \qquad$ in Ω,

where Ω is a bounded domain in $\boldsymbol{R}^n$ with smooth boundary $\partial\Omega$, f is Hölder-continuous on $(0,\infty)\times\Omega$ and $f\geqq 0$, $\alpha\in C^2(\partial\Omega)$ satisfies $0\leqq\alpha<1$ or $\alpha\equiv 1$ on $\partial\Omega$, $u_0\in C^0(\bar{\Omega})$, and $u_0|_{\partial\Omega}=0$ in the case $\alpha\equiv 1$, M is a given positive number.

Note that the existence of a global solution is highly restricted in the case of the equation of evolution:

(1.4) $\quad U_t=\Delta U+e^U+f$

associated with (1.2) and (1.3) (see Fujita [1]).

We shall also show the existence of a solution u_∞ of the following stationary variational inequality:

(1.5) $\quad (-\Delta u_\infty-e^{u_\infty}-g,\ v-u_\infty)_{L^2(\Omega)}\geqq 0$

for any $v\in L^2(\Omega)$ satisfying $0\leqq v\leqq M$ a.e. in Ω, and $0\leqq u_\infty\leqq M$, where g is Hölder-continuous in Ω and $g\geqq 0$, and the boundary condition

(1.6) $\quad \alpha u_\infty+(1-\alpha)(\partial u_\infty/\partial\boldsymbol{n})=0.$

As in (1.4), we cannot always expect the existence of a sclution U_∞ of the equation (1.7) under the boundary condition (1.8):

(1.7) $\quad \Delta U_\infty+e^{U_\infty}+g=0 \qquad$ in Ω,

(1.8) $\quad \alpha U_\infty+(1-\alpha)(\partial U_\infty/\partial\boldsymbol{n})=0 \qquad$ on $\partial\Omega$.

If f is equal to g which is independent of the time t, then we can show that the solution $u(t,\cdot)$ of (1.1)-(1.2)-(1.3) converges to the solution u_∞ of (1.5)-(1.6) as t tends to ∞.

§2. **Statement of Theorems.**

Theorem 1. *Under the conditions stated in* §1, *there exists one and only one solution* $u\equiv u(t,\cdot)$ *of* (1.1)-(1.2)-(1.3) *which satisfies the following conditions* (2.1), (2.2), (2.3) *and* (2.4):

(2.1) $\quad u\in C^0([0,T]\times\bar{\Omega})$,

(2.2) $\quad D_x u\in C^0([\delta,T]\times\bar{\Omega})$, (2.3) *$u$ is differentiable at* a.e. $t\in(\delta, T)$ *as an* $L^2(\Omega)$*-valued function on* (δ, T), *and* $u_t \in L^2(\delta, T; L^2(\Omega))$,

(2.4) $$\Delta u \in L^2(\delta, T; L^2(\Omega)),$$

where δ *and* T *are arbitrary positive numbers such that* $\delta<T$, $D_x u$ *denotes any first order partial derivative in space variables.*

Theorem 2. *Let* u_i *be the solution of* (1.1)-(1.2)-(1.3) *corresponding to* f_i *and* u_{0i} *for* $i=1, 2$. *If* $f_1 \geqq f_2$ *and* $u_{01}\geqq u_{02}$, *then* $u_1 \geqq u_2$ *on* $[0, \infty)\times\bar{\Omega}$.

Theorem 3. *Under the conditions stated in* § 1, *there exists a solution* u_∞ *of* (1.5)-(1.6) *which satisfies*

(2.5) $$u_\infty \in C^1(\bar{\Omega})\cap H^2(\Omega).$$

If $\alpha\equiv 1$ *and diameter of* Ω *is sufficiently small* (*for example* $\operatorname{diam}\Omega < (2ne^{-M})^{1/2}$), *then the solution is unique.*

Theorem 4. *Let* $u\equiv u(t, \cdot)$ *be the solution of* (1.1)-(1.2)-(1.3) *satisfying* (2.1), (2.2), (2.3) *and* (2.4) *where we assume* $f(t, \cdot)\equiv g(\cdot)$. *Then the following* (i) *and* (ii) *hold*:

(i) *If* $\alpha\equiv 0$, *then there exist* $t_*>0$ *and a nonnegative, continuous and monotone decreasing function* $c(t)$ *satisfying* $c(t)=0$ *for any* $t>t_*$ *such that* $u(t, \cdot)$ *satisfies*

(2.6) $$\|u(t, \cdot)-M\|_{L^\infty(\Omega)}\leqq c(t) \qquad \text{for any } t\geqq 0.$$

(ii) *If* $\alpha\equiv 1$ *and* $\operatorname{diam}\Omega$ *is sufficiently small* (*as stated in Theorem* 3), *then there exist positive numbers* δ *and* C *such that*

(2.7) $$\|u(t, \cdot)-u_\infty\|_{L^2(\Omega)}\leqq Ce^{-\delta t} \qquad \text{for any } t\geqq 0,$$

where u_∞ *is the solution of the stationary variational inequality* (1.5)-(1.6) *under the Dirichlet boundary condition.*

§ 3. Outline of the proofs of theorems.

Proof of Theorem 1. We use so-called penalty method. We shall prove this theorem in the case; $\alpha\equiv 1$, $f\equiv 0$ and $u_0<M$. The proof of general case may be performed analogously. For any positive number $\varepsilon<e^{-M}/2$, we define the mapping $\beta_\varepsilon : \boldsymbol{R}^1\to\boldsymbol{R}^1$ as follows:

$$\beta_\varepsilon(\lambda)=\frac{1}{\varepsilon}(e^{\lambda-M}-1)_+=\operatorname{Max}\left\{\frac{1}{\varepsilon}(e^{\lambda-M}-1), 0\right\}.$$

We also fix a C^1-mapping $\gamma : \boldsymbol{R}^1\to\boldsymbol{R}^1$ satisfying the following condition:

$\gamma(\cdot)$ is the identity mapping on $[0, \infty)$, and is bounded and negative on $(-\infty, 0)$.

Let $U(t, x, y)$ be the fundamental solution of the parabolic equation $u_t=\Delta u$ in Ω under the Dirichlet boundary condition. We construct "approximate functions" u_ε by solving a Volterra type integral equation by iteration as follows:

(3.1) $$\begin{cases} u_0(t, x)=u_0(x), \\ u_{n+1}(t, x)=\displaystyle\int_\Omega U(t, x, y)u_0(y)dy \\ \qquad\qquad\qquad +\displaystyle\int_0^t\int_\Omega U(t-\tau, x, y)\gamma(e^{u_n}-\beta_\varepsilon(u_n))dyd\tau \\ (n=0, 1, 2, \cdots). \end{cases}$$

Because of the boundedness of $\gamma(e^{\lambda}-\beta_{\varepsilon}(\lambda))$ on $\boldsymbol{R}^1$, $\|u_n\|_{L^\infty((0,T)\times\Omega)}$ is bounded uniformly in n. By standard argument, we can show that u_n converges to a function u_ε uniformly on $[0, T]\times\bar{\Omega}$ which satisfies

$$(3.2)\qquad \begin{cases} u_\varepsilon(t,x)=\int_\Omega U(t,x,y)u_0(y)dy \\ \qquad\qquad +\int_0^t\int_\Omega U(t-\tau,x,y)\gamma(e^{u_\varepsilon}-\beta_\varepsilon(u_\varepsilon))dyd\tau \end{cases}$$

on $(0, T]\times\bar{\Omega}$.

Accordingly, u_ε satisfies

$$(3.3)\qquad \begin{cases} (u_\varepsilon)_t=\Delta u_\varepsilon+\gamma(e^{u_\varepsilon}-\beta_\varepsilon(u_\varepsilon)) \\ u_\varepsilon|_{[0,T]\times\partial\Omega}=0. \end{cases}$$

We may also show $0\leqq u_\varepsilon\leqq\lambda_\varepsilon$ on $[0, T]\times\bar{\Omega}$, where λ_ε is a unique root of $e^\lambda=\beta_\varepsilon(\lambda)$. So we may replace $\gamma(e^{u_\varepsilon}-\beta_\varepsilon(u_\varepsilon))$ in (3.4) by $e^{u_\varepsilon}-\beta_\varepsilon(u_\varepsilon)$, and we may conclude that u_ε satisfies

$$(3.4)\qquad \begin{cases} (u_\varepsilon)_t=\Delta u_\varepsilon+e^{u_\varepsilon}-\beta_\varepsilon(u_\varepsilon) \\ u_\varepsilon|_{[0,T]\times\partial\Omega}=0. \end{cases}$$

Next we choose a positive number η so small that $\|u_\varepsilon(t,\cdot)\|_{L^\infty((0,\eta)\times\Omega)}<M$. Then on $[0,\eta]\times\bar{\Omega}$, u_ε is the solution of $U_t=\Delta U+e^U$, and u_ε and $D_x u_\varepsilon$ are uniformly bounded and equicontinuous on $[\eta, T]\times\bar{\Omega}$; this fact follows from (3.2) and the property of the fundamental solution $U(t, x, y)$. Applying the Ascoli-Arzelà Theorem, there exist a sequence $\{\varepsilon_n\}\downarrow 0$ and a function u such that u_{ε_n} (resp. $D_x u_{\varepsilon_n}$) converges to u (resp. $D_x u$) uniformly on $[\eta, T]\times\bar{\Omega}$. Thus we have proved (2.1) and (2.2). Moreover there exists a nonnegative function $B\in L^2((0,T)\times\Omega)$ such that $\beta_{\varepsilon_n}(u_{\varepsilon_n})$ converges to B weakly; this fact follows from the uniform boundedness of $\|\beta_\varepsilon(u_\varepsilon)\|_{L^\infty((\eta,T)\times\Omega)}$. We can also show that $\{(u_\varepsilon)_t\}_{0<\varepsilon<e^{-M/2}}$ is bounded in $L^2(\eta, T; L^2(\Omega))$. So (2.3) holds, and accordingly (2.4) and the following equation holds:

$$(3.5)\qquad u_t=\Delta u+e^u-B \qquad \text{in } L^2(\eta, T; L^2(\Omega)).$$

To show (1.1), we take any t_1, t_2 $(\eta\leqq t_1<t_2\leqq T)$ and $v\in L^2(\eta, T; L^2(\Omega))$ satisfying $0\leqq v\leqq M$. Then

$$(3.6)\qquad \int_{t_1}^{t_2}(u_t-\Delta u-e^u, v-u)_{L^2(\Omega)}d\tau=\int_{t_1}^{t_2}(-B, v-u)_{L^2(\Omega)}d\tau$$
$$=\int_{t_1}^{t_2}(-B, v-M)_{L^2(\Omega)}d\tau+\int_{t_1}^{t_2}(-B, M-u)_{L^2(\Omega)}d\tau=\mathrm{I}+\mathrm{II}.$$

That $\mathrm{I}\geqq 0$ is clear. That $\mathrm{II}=0$ follows from the estimate of λ_ε and $\|\beta_\varepsilon(u_\varepsilon)\|_{L^\infty((0,T)\times\Omega)}$. Thus we have proved the existence of a solution.

If there exists another solution $\hat{u}$ which satisfies (1.1)-(1.2)-(1.3) and (2.1), (2.2), (2.3) and (2.4), then the difference $w=u-\hat{u}$ satisfies

$$\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)}\leqq 2e^M\|w(t)\|^2_{L^2(\Omega)},$$

and accordingly $\|w(t)\|^2_{L^2(\Omega)}\leqq\|w(\eta)\|^2_{L^2(\Omega)}\times e^{2e^M t}=0$.

Proof of Theorem 2. Let $u_{i\varepsilon}$ be the approximate function of u_i stated in the proof of Theorem 1 $(i=1,2)$. Then we can show $u_{1\varepsilon}\geqq u_{2\varepsilon}$. Taking limit along the common subsequence $\{\varepsilon_n\}\downarrow 0$, we obtain $u_1\geqq u_2$.

Proof of Theorem 3. To construct approximate functions, we define the operator $\Gamma : C^0(\bar{\Omega}) \to C^0(\bar{\Omega})$ as follows;

$$(\Gamma u)(x) = \int_\Omega G(x, y)\gamma(e^u - \beta_\varepsilon(u))dy, \tag{3.8}$$

where G is the Green function under the given boundary condition. Then Γ is a compact operator in $C^0(\bar{\Omega})$. Hence, using Schauder's fixed point Theorem, we can define approximate functions. Using the argument used in the proof of Theorem 1, we can prove the existence of a solution.

If there exists another solution $\hat{u}_\infty$, the difference $w = u_\infty - \hat{u}_\infty$ satisfies $-\|\nabla w\|^2_{L^2(\Omega)} + e^M \|w\|^2_{L^2(\Omega)} \geqq 0$. If diam Ω is so small as stated in Theorem 3, $\|w\|^2_{L^2(\Omega)} = 0$ follows from the Poincaré inequality.

Proof of Theorem 4. We may also show that the difference $w(t) = u(t) - u_\infty$ satisfies $(1/2)(d/dt)\|w(t)\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)} \leqq e^M \|w\|^2_{L^2(\Omega)}$. If diam Ω is so small as stated in Theorem 3, Theorem 4-(i) follows from the Poincaré inequality. (ii) is proved by using Theorem 2.

Remark. It is not very difficult to extend our results to the case of more general C^1-nonlinear terms. Of course the u^m-type nonlinear term can be treated easily. Details will be published elsewhere.

References

[1] H. Fujita: On the nonlinear equations $\Delta u + e^u = 0$ and $v_t = \Delta v + e^v$. Bull. Amer. Math. Soc., **75**, no. 1, 132–135 (1969).

[2] S. Itô: Fundamental solutions of parabolic differential equations and boundary value problems. Japan. J. Math., **27**, 55–102 (1957).