46. Continuity Theorem for Non-linear Integral Functionals and Aumann-Perles' Variational Problem

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1. Introduction. Let $(T, \mathcal{E}, \mu)$ be a measure space and assume that a couple of functions $u: T\times \boldsymbol{R}^l \to \boldsymbol{R}$ and $g: T\times \boldsymbol{R}^l \to \boldsymbol{R}^k$, as well as a vector $\omega \in \boldsymbol{R}^k$ are given. Consider the well-known Aumann-Perles' variational problem formulated as follows:

$$\text{(P)} \qquad \begin{cases} \underset{x}{\mathit{Maximize}} \int_T u(t, x(t))d\mu \\ \qquad \textit{subject to} \\ \qquad\qquad \int_T g(t, x(t))d\mu \leqq \omega. \end{cases}$$

The existence of optimal solutions for (P) has been investigated by Artstein [2], Aumann-Perles [3], Berliocchi-Lasry [5], Maruyama [8] and others. In this paper, we shall present an alternative approach to the existence problem, being based upon the continuity theorem for non-linear integral functionals due to Berkovitz [4] and Ioffe [6].

2. Continuity and compactness of level sets for non-linear integral functionals. In the proof of our main theorem discussed in the next section, we shall effectively make use of a couple of results in non-linear functional analysis. We had better summarize them here for the sake of readers' convenience.

Continuity Theorem (Berkovitz [4], Ioffe [6]). *Let $(T, \mathcal{E}, \mu)$ be a non-atomic complete finite measure space and $f: T\times \boldsymbol{R}^l \times \boldsymbol{R}^k \to \bar{\boldsymbol{R}}$ be a convex normal integrand. Define a non-linear functional $J: L^p(T, \boldsymbol{R}^l)\times L^q(T, \boldsymbol{R}^k) \to \bar{\boldsymbol{R}} (p, q \geqq 1)$ by*

$$J(x, y) = \int_T f(t, x(t), y(t))d\mu.$$

If there exist some $a \in L^{q'}(T, \boldsymbol{R}^k)$ (where $1/q + 1/q' = 1$) and $b \in L^1(T, \boldsymbol{R})$ such that

$$f(t, x, y) \geqq \langle a(t), y\rangle + b(t)$$
$$(\langle \cdot, \cdot\rangle \textit{ stands for the inner product})$$

for all $(t, x, y) \in T\times \boldsymbol{R}^l \times \boldsymbol{R}^k$, then J is sequentially lower semi-continuous with respect to the strong topology on $L^p(T, \boldsymbol{R}^l)$ and the weak topology on $L^q(T, \boldsymbol{R}^l)$.

Compactness Theorem (Ioffe-Tihomirov [7]). *Let $(T, \mathcal{E}, \mu)$ be a finite measure space and $f: T\times \boldsymbol{R}^l \to \bar{\boldsymbol{R}}$ be $(\mathcal{E}\otimes\mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\bar{\boldsymbol{R}}))$-measurable, where $\mathcal{B}(\cdot)$ stands for the Borel σ-field on $(\cdot)$. If f satisfies the growth condition:*

$$\mathrm{Dom}\int_T |f^*(t, y)|\, d\mu = \boldsymbol{R}^l$$

(where $f^*(t, \cdot)$ denotes the Young-Fenchel transform of $f: x \mapsto f(t, x)$ for any fixed $t \in T$), then the set

$$F_c = \left\{ x \in L^1(T, \boldsymbol{R}^l) \,\middle|\, \int_T f(t, x(t))\, d\mu \leqq c \right\}$$

is weakly relatively compact in $L^1(T, \boldsymbol{R}^l)$ for any $c \in \boldsymbol{R}$.

For systematic and extensive studies on these topics, see Maruyama [9] Chap. 9.

3. Main Theorem. We shall now turn to the Aumann-Perles' problem (P).

Assumption 1. $(T, \mathcal{E}, \mu)$ is a non-atomic, complete finite measure space.

Assumption 2. u satisfies the following conditions.

(1) u is $(\mathcal{E} \otimes \mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\boldsymbol{R}))$-measurable.

(2) The function $x \mapsto u(t, x)$ is upper semi-continuous and concave for any fixed $t \in T$.

(3) There exist some $a \in L^\infty(T, \boldsymbol{R}^l)$ and $b \in L^1(T, \boldsymbol{R})$ such that

$$u(t, x) \leqq \langle a(t), x \rangle + b(t)$$

for all $(t, x) \in T \times \boldsymbol{R}^l$.

(4) $\int_T u(t, x(t))\, d\mu > -\infty$

for all $x \in L^1(T, \boldsymbol{R}^l)$.

Assumption 3. $g \equiv (g^{(1)}, g^{(2)}, \cdots, g^{(k)})$ satisfies the following conditions.

(1) $g^{(i)}$ is $(\mathcal{E} \otimes \mathcal{B}(\boldsymbol{R}^l), \mathcal{B}(\boldsymbol{R}))$-measurable.

(2) The function $x \mapsto g^{(i)}(t, x)$ is lower semi-continuous and convex for any fixed $t \in T$.

(3) There exist some $c \in L^\infty(T, \boldsymbol{R}^l)$ and $d \in L^1(T, \boldsymbol{R})$ such that

$$g^{(i)}(t, x) \geqq \langle c(t), x \rangle + d(t)$$

for all $(t, x) \in T \times \boldsymbol{R}^l$.

(4) $g^{(i)}$ satisfies the growth condition:

$$\mathrm{Dom}\int_T |g^{(i)*}(t, y)|\, d\mu = \boldsymbol{R}^l.$$

Theorem. *Under Assumptions* 1~3, *our problem* (P) *has an optimal solution in $L^1(T, \boldsymbol{R}^l)$.*

Proof. According to the Continuity Theorem, Assumptions 1~2 imply that the integral functional

$$J: x \longmapsto \int_T u(t, x(t))\, d\mu$$

is sequentially upper semi-continuous on $L^1(T, \boldsymbol{R}^l)$ with respect to the weak topology.

And Assumption 3 assures, by the Compactness Theorem, that the set

$$F_\omega = \left\{ x \in L^1(T, \boldsymbol{R}^l) \,\middle|\, \int_T g(t, x(t))\, d\mu \leqq \omega \right\}$$

is weakly relatively compact in $L^1(T, \boldsymbol{R}^l)$. Hence F_ω is L^1-bounded. Thus we obtain, by Assumption 2(3), that

$$-\infty<\gamma\equiv\sup_{x\in F_\omega}J(x)\leqq\|a\|_\infty\cdot\sup_{x\in F_\omega}\|x\|_1+\|b\|_1\equiv C<\infty$$

($-\infty<\gamma$ comes from Assumption 2(4)).

Let $\{x_n\}$ be a sequence in F_ω such that

$$\lim_{n\to\infty}J(x_n)=\gamma.$$

Since F_ω is weakly relatively compact, $\{x_n\}$ has a convergent subsequence. Without loss of generality, we may assume that

$$w\text{-}\lim_{n\to\infty}x_n=x^*\in L^1(T,\boldsymbol{R}^l).$$

We can easily verify that $x^*\in F_\omega$ as follows. Again by the Continuity Theorem, Assumptions 1 and 3 imply that the integral functional

$$I_i:x\longmapsto\int_T g^{(i)}(t,x(t))d\mu$$

is sequentially lower semi-continuous on $L^1(T,\boldsymbol{R}^l)$ with respect to the weak topology. Hence

$$\int_T g^{(i)}(t,x^*(t))d\mu\leqq\liminf_n\int_T g^{(i)}(t,x_n(t))d\mu\leqq\omega^{(i)},$$

from which we can conclude that $x^*\in F_\omega$.

Finally, by the sequential upper semi-continuity of J, we must have

$$J(x^*)\geqq\limsup_n J(x_n)\equiv\gamma.$$

On the other hand, it is obvious that $\gamma\geqq J(x^*)$. Hence $J(x^*)=\gamma$, which means that x^* is an optimal solution for (P). Q.E.D.

Essentially the same technique can be applied to the existence proof for the Arkin-Levin's variational problem ([1]). For the details, see Maruyama [9] Chap. 9.

References

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