45. Representations of Hecke Algebras on Virtual Character Modules of a Semisimple Lie Group*)

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§1. **Introduction.** The purpose of this note is to present, together with some further consequences, the main results of our Thesis [6] to which we shall leave detailed descriptions. Let G be a connected semisimple Lie group with finite centre, which is assumed to be "acceptable", i.e., satisfying certain natural conditions for technical reasons. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_C$ its complexification. We denote by $U(\mathfrak{g}_C)$ the enveloping algebra of $\mathfrak{g}_C$ and by $\mathfrak{Z}$ the centre of $U(\mathfrak{g}_C)$.

Let $\mathrm{Car}(G)$ be the set of all the conjugacy classes of Cartan subgroups of G. Then $\sharp\mathrm{Car}(G)$ is finite. Take $[H]\in\mathrm{Car}(G)$, where $[H]$ means the conjugacy class of a Cartan subgroup H. We fix H as a representative of the class. Let $\mathfrak{h}$ and $\mathfrak{h}_C$ be the Lie algebra of H and its complexification respectively. Put $W=W(\mathfrak{g}_C,\mathfrak{h}_C)$, the complex Weyl group. For each $\lambda\in\mathfrak{h}_C^*$, we define a subgroup $W_H(\lambda)$ and a subset $W_H^{\sim}(\lambda)$ of W as in [4, pp. 724–725]. We call $W_H(\lambda)$ the *integral Weyl group* for H and λ. We also define the fixed subgroup of λ as $W_\lambda=\{w\in W\,|\,w\lambda=\lambda\}$.

For each irreducible admissible representation $(\pi,\mathfrak{H})$ of G on a Hilbert space $\mathfrak{H}$, there corresponds to $\chi\in\mathrm{Hom}_{alg}(\mathfrak{Z},C)$ so-called infinitesimal character of π. By Harish-Chandra homomorphism, $\mathfrak{Z}$ is isomorphic to $U(\mathfrak{h}_C)^W$ as an algebra, where $U(\mathfrak{h}_C)^W$ denotes the set of all the W-fixed elements in $U(\mathfrak{h}_C)$. Since $\mathrm{Hom}_{alg}(U(\mathfrak{h}_C)^W,C)\simeq\mathfrak{h}_C^*/W$, there exists $\lambda\in\mathfrak{h}_C^*$ which naturally defines χ. We denote this as $\chi=\chi_\lambda$. Remark that $w\lambda(w\in W)$ also defines χ. Let $\mathrm{Mod}(\chi)$ be the set of irreducible admissible representations of G with infinitesimal character χ. For $\pi\in\mathrm{Mod}(\chi)$, one can define the character θ_π which is a constant coefficient invariant eigendistribution on G [5]. Put

$$V(\chi)=\langle\theta_\pi\,|\,\pi\in\mathrm{Mod}(\chi)\rangle\qquad\text{(generated as a vector space over } C).$$

Then $V(\chi)$ is finite dimensional. We can define subspaces $V_H(\chi)$ ($[H]\in\mathrm{Car}(G)$), denoted by $V_H(\lambda)$ in [4], and we have the direct sum decomposition [4, p. 726]

$$V(\chi)=\bigoplus_{[H]\in\mathrm{Car}(G)}V_H(\chi).$$

§2. **Representations of Hecke algebras $\mathcal{H}(W_H(\lambda),W_\lambda)$ on $V_H(\chi)$.** Fix a Cartan subgroup H. In the preceding paper [4, §3], we define a repre-

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sentation τ of the integral Weyl group $W_H(\lambda)$ on the space $V_H(\chi)$ in case that λ is regular. This representation τ seems to be concerned with Joseph's Goldie rank representations which arises from the study of primitive ideals of $U(\mathfrak{g}_C)$.

If the infinitesimal character λ is singular, there isn't any representation of Weyl groups on the virtual character modules. However, replacing Weyl groups by their Hecke algebras, we can again construct natural representations on the virtual character modules. More precisely, for general λ, we define a representation σ of Hecke algebras $\mathcal{H}(W_H(\lambda), W_\lambda)$ on the space $V_H(\chi)$, which is naturally identified with τ if λ is regular.

Let $\{H_i \mid 0 \leqq i \leqq l\}$ be the system of representatives of conjugacy classes of connected components of H under G. We take $H_0 = \exp \mathfrak{h}$, the neutral component of H. For a subgroup B of G and a subset D of G (or of $\mathfrak{g}$), we define $W(B; D) = N_B(D)/Z_B(D)$, where $N_B(D)$ denotes the normalizer of D in B and $Z_B(D)$ the centralizer. We assume that λ satisfies the following assumption.

Assumption 2.1. There exists a subset $\{a_i \mid 0 \leqq i \leqq l\}$ of H such that

(0) $\quad a_i \in H_i \ (0 \leqq i \leqq l)$ and $a_0 = e$.

(1) $\quad \xi_{t\lambda}(a_i^{-1} s a_i) = 1$ for any $t \in W_H^{\sim}(\lambda)$ and $s \in W(G; H_i)$.

Here $\xi_{\lambda'}$ is a character of H_0 defined by $\xi_{\lambda'}(\exp x) = \exp \lambda'(x)$ $(x \in \mathfrak{h})$.

We define an analytic function $\zeta(i, t; *)$ $(0 \leqq i \leqq l, t \in W_H^{\sim}(\lambda))$ on H as in [4, Proposition 1.7] and put

$$\mathfrak{C}(H; \lambda) = \langle \zeta(i, t; *) \mid 0 \leqq i \leqq l, t \in W_H^{\sim}(\lambda) \rangle / \boldsymbol{C}.$$

Then we have $V_H(\chi) \simeq \mathfrak{C}(H; \lambda)$ as vector spaces. Let $\boldsymbol{T}: \mathfrak{C}(H; \lambda) \xrightarrow{\sim} V_H(\chi)$ be the isomorphism given by T. Hirai [2, § 3].

Since W_λ is a subgroup of $W_H(\lambda)$, we can form the Hecke algebra $\mathcal{H} = \mathcal{H}(W_H(\lambda), W_\lambda)$ as in [3]. Put

$$e_\lambda = (\sharp W_\lambda)^{-1} \sum_{s \in W_\lambda} s \in \boldsymbol{C}[W_H(\lambda)],$$

where $\boldsymbol{C}[W_H(\lambda)]$ means the group ring of $W_H(\lambda)$. Then we have $\mathcal{H}(W_H(\lambda), W_\lambda) \simeq e_\lambda \boldsymbol{C}[W_H(\lambda)] e_\lambda$ canonically as algebras. Here we always use this identification.

Take $e_\lambda w e_\lambda \in \mathcal{H}$. We define an action σ of $\mathcal{H}$ on $V_H(\chi)$ as

$$\sigma(e_\lambda w e_\lambda) \boldsymbol{T}(\zeta(i, t; *)) = (\sharp W_\lambda)^{-1} \sum_{s \in W_\lambda} \boldsymbol{T}(\zeta(i, tsw^{-1}; *)).$$

Theorem 2.2. *The action σ of the Hecke algebra $\mathcal{H} = \mathcal{H}(W_H(\lambda), W_\lambda)$ given above defines a representation of $\mathcal{H}$.*

§ 3. Decomposition of the representations. Let A be a finite group and B a subgroup of A. Put $e_B = (\sharp B)^{-1} \sum_{b \in B} b$. Then it holds that $\mathcal{H}(A, B) \simeq e_B \boldsymbol{C}[A] e_B$ and, hence, we identify these two algebras. Take an irreducible representation (τ, V) of A. We make a representation $\mathrm{Red}_B^A \tau$ of $\mathcal{H}(A, B)$ from (τ, V) as follows. The representation space of $\mathrm{Red}_B^A \tau$ is a subspace $\tau(e_B)V$ of V consisting of all the B-fixed vectors. The action of $\mathcal{H}(A, B)$ is defined by $\tau|_{\mathcal{H}(A,B)}$, where we extend τ to the representation of the group ring $\boldsymbol{C}[A]$. This representation $\mathrm{Red}_B^A \tau$ is irreducible (but possibly zero).

Return to our original notations. Let $\varepsilon(a_i\,;s)$ $(0\leqq i\leqq l,\ s\in W(G\,;H_i))$ be a character of $W(G\,;H_i)$ defined as in the formula (2.3) in [1, p. 36]. We choose a complete system of representatives Γ of the double coset space $W(G\,;H_i)\backslash W_H^{\sim}(\lambda)/W_H(\lambda)$. For each $\gamma\in\Gamma$ and $0\leqq i\leqq l$, we put

$$W(i,\gamma)=W_H(\lambda)\cap\gamma^{-1}W(G\,;H_i)\gamma,$$

$$\varepsilon^{i,\gamma}(w)=\varepsilon(a_i\,;\gamma w\gamma^{-1})\qquad(w\in W(i,\gamma)).$$

Then $\varepsilon^{i,\gamma}$ is a character of the subgroup $W(i,\gamma)$ of $W_H(\lambda)$.

Theorem 3.1. *The representation σ of the Hecke algebra $\mathcal{H}$ on $V_H(\chi)$ given in Theorem 2.2 is decomposed as follows*:

$$(\sigma,\ V_H(\chi))\simeq\sum_{i=0}^{l}{}^{\oplus}\sum_{\gamma\in\Gamma}{}^{\oplus}\operatorname{Red}_{W_\lambda}^{W_H(\lambda)}\operatorname{Ind}_{W(i,\gamma)}^{W_H(\lambda)}\varepsilon^{i,\gamma}.$$

In particular, if $W_H(\lambda)=W$ for any $[H]\in\operatorname{Car}(G)$, we have a representation $(\sigma, V(\chi))$ of $\mathcal{H}(W, W_\lambda)$ summing up $(\sigma, V_H(\chi))$ through $[H]\in\operatorname{Car}(G)$. In this case we have

Corollary 3.2. *The representation $(\sigma, V(\chi))$ of $\mathcal{H}(W, W_\lambda)$ is decomposed as*

$$(\sigma,\ V(\chi))\simeq\sum_{[H]\in\operatorname{Car}(G)}{}^{\oplus}\ \sum_{i=0}^{l}{}^{\oplus}\operatorname{Red}_{W_\lambda}^{W}\operatorname{Ind}_{W(G;H_i)}^{W}\varepsilon(a_i\,;*).$$

§ 4. Translation functors and the representations of the Hecke algebra. Choose a set of positive roots for $(\mathfrak{g}_C, \mathfrak{h}_C)$ such that $\operatorname{Re}\langle\lambda,\alpha\rangle\geqq 0$. Take $\mu\in\mathfrak{h}_C^*$ such that $\lambda+\mu$ is dominant regular and μ itself belongs to the weight lattice. Let $\varphi=\varphi_{\lambda+\mu}^{\lambda}$ and $\psi=\psi_{\lambda}^{\lambda+\mu}$ be Zuckerman's translation functors (see [7]). We put $\chi=\chi_\lambda$, $\chi'=\chi_{\lambda+\mu}$.

Lemma 4.1. *The integral Weyl groups $W_H(\lambda)$ and $W_H(\lambda+\mu)$ are equal.*

Theorem 4.2. *The representation $(\sigma, V_H(\chi))$ of $\mathcal{H}(W_H(\lambda), W_\lambda)$ has a following expression, using the representation $(\tau, V_H(\chi'))$ of $W_H(\lambda)$ and the translation functors*:

$$\sigma(e_\lambda s e_\lambda)v=(\sharp W_\lambda)^{-1}\psi\circ\tau(s)\circ\varphi(v)\qquad(v\in V_H(\chi),\ s\in W_H(\lambda)).$$

This theorem tells us that the representation $(\sigma, V_H(\chi))$ is a "limit" of $(\tau, V_H(\chi'))$. Translation functors appear naturally on studying such as classification theory of representations, existence of some "limit" representations and so on. Hence this theorem might be useful for considering such directions.

§ 5. The case of complex Lie groups. Let G be a connected and simply connected complex semisimple Lie group. Then G has the unique conjugacy class of Cartan subgroups. Take a Cartan subgroup H of G, and put $W=W(\mathfrak{g},\mathfrak{h})$ the Weyl group of $(\mathfrak{g},\mathfrak{h})$. We have $W=W(G\,;H)$ and the Weyl group of $(\mathfrak{g}_C, \mathfrak{h}_C)$ is isomorphic to $W\times W$. $W=W(G\,;H)$ is imbeded into $W\times W$ as a diagonal subgroup: $W\ni w\to(w,w)\in\Delta W\subset W\times W$. Let P be the weight lattice of $(\mathfrak{g}_C, \mathfrak{h}_C)$. Then we have the following result.

(1) If $\lambda\in P$ is regular, we have the representation $(\tau, V_H(\chi_\lambda))$ of $W_H(\lambda)=W\times W$. This representation is equivalent to $\operatorname{Ind}_{\Delta W}^{W\times W}$(trivial rep.) by Theorem 3.1. This is exactly the two-sided regular representation of $W\times W$ on $\boldsymbol{C}[W]$.

(2) Let $\lambda \in P$ be singular. Then the fixed subgroup $(W\times W)_\lambda$ of λ is represented as $(W\times W)_\lambda = W_{\lambda_1}\times W_{\lambda_2}$ naturally, where λ_1 and λ_2 denote the anti-holomorphic and holomorphic part of λ. In this case, we have the representation $(\sigma, V_H(\chi_\lambda))$ of $\mathcal{H}(W\times W, W_{\lambda_1}\times W_{\lambda_2})\cong\mathcal{H}(W, W_{\lambda_1})\otimes\mathcal{H}(W, W_{\lambda_2})$. σ is isomorphic to the representation $(\sigma', e_{\lambda_1}\boldsymbol{C}[W]e_{\lambda_2})$, where σ' is given by

$$\sigma'(e_{\lambda_1}s_1e_{\lambda_1}\otimes e_{\lambda_2}s_2e_{\lambda_2})v = e_{\lambda_1}s_1vs_2^{-1}e_{\lambda_2} \quad (s_1, s_2 \in W \text{ and } v \in e_{\lambda_1}\boldsymbol{C}[W]e_{\lambda_2}).$$

For a general $\lambda \in \mathfrak{h}_{\boldsymbol{C}}^*$, we can get the similar results.

Proposition 5.1. *Let $\lambda \in P$ and $(W\times W)_\lambda = W_{\lambda_1}\times W_{\lambda_2}$ be as above. Then we have*

$$\dim V(\chi_\lambda) = \dim e_{\lambda_1}\boldsymbol{C}[W]e_{\lambda_2} = \dim \mathrm{Hom}_W(\boldsymbol{C}[W]e_{\lambda_1}, \boldsymbol{C}[W]e_{\lambda_2}).$$

This dimension is equal to the number of infinitesimal equivalence classes of irreducible admissible representations of G with infinitesimal character λ.

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