44. On the Kazhdan-Lusztig Conjecture for Kac-Moody Algebras

By Kiyokazu SUTO
Department of Mathematics, Kyoto University

(Communicated by Shokichi IYANAGA, M. J. A., April 14, 1986)

Let $\mathfrak{g}$ be a Kac-Moody algebra over $\boldsymbol{C}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\rho \in \mathfrak{h}^*$ be an element which takes the value 1 on each simple coroot and W the Weyl group of $\mathfrak{g}$. We denote by $M(\lambda)$ and $L(\lambda)$ the Verma module with highest weight $\lambda \in \mathfrak{h}^*$ and the unique irreducible quotient of $M(\lambda)$ respectively.

If the generalized Cartan matrix (GCM) corresponding to $\mathfrak{g}$ is symmetrizable, then it was proved in [3] analogously to the (finite-dimensional) complex semisimple case that for any dominant integral element $\lambda \in \mathfrak{h}^*$ and $y \in W$, all the irreducible subquotients of $M(y(\lambda+\rho)-\rho)$ are $L(w(\lambda+\rho)-\rho)$ with $w \in W$ such that $w \geqq y$, and that multiplicities $\operatorname{mtp}(y, w)=[M(y(\lambda+\rho)-\rho) : L(w(\lambda+\rho)-\rho)]$ are independent of λ. Here $\leqq$ is the standard partial order on the Coxeter group W in which the unit is the smallest element. Note that $\operatorname{mtp}(y, w)=0$ if $y \nleqq w$.

In case where $\mathfrak{g}$ is finite-dimensional, the following proposition on these multiplicities, well-known as the Kazhdan-Lusztig conjecture, was proved in [2] and in [1] independently.

Theorem A [5, Conjecture 1.5]. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Under the same notations as above,*

$$\operatorname{mtp}(y, w)=P_{y,w}(1)$$

holds for all $y, w \in W$ such that $y \leqq w$, where $P_{y,w}$ are the Kazhdan-Lusztig polynomials for the Coxeter group W.

Kazhdan-Lusztig polynomials were introduced in [5], related to a base change of Hecke algebras of Coxeter groups, and there were given inductive formulas to compute these polynomials.

Deodhar, Gabber and Kac conjectured in [3] that the same result as Theorem A holds for Kac-Moody algebras of infinite-dimension as follows.

Conjecture B [3, Conjecture 5.16]. Let $\mathfrak{g}$ be a Kac-Moody algebra corresponding to a symmetrizable GCM. Then

$$\operatorname{mtp}(y, w)=P_{y,w}(1)$$

holds for all $y, w \in W$ such that $y \leqq w$.

In this paper, we prove this conjecture is true for certain pairs (y, w), by reducing it to the finite-dimensional case. Even when the GCM is not symmetrizable, this holds for $\lambda=0$, the most important case. We give further a branching rule of Verma modules over a non-twisted affine Lie algebra with respect to a certain subalgebra.

§1. Some properties of Kac-Moody algebras. Let Δ be the root system of $\mathfrak{g}$, and $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ the root space decomposition. Let $\Pi=\{\alpha_1, \cdots, \alpha_n\}$ (resp. $\Pi^\vee=\{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$) be the set of simple roots (resp. simple coroots). See [4, §1] for detail. We put $Q_+=\sum_{1\leqq i\leqq n}\boldsymbol{Z}_{\geqq 0}\alpha_i$ and define a partial order on $\mathfrak{h}^*$ by

$$\lambda\geqq\mu \text{ if and only if } \lambda-\mu\in Q_+ \quad (\lambda, \mu\in\mathfrak{h}^*).$$

We denote by Δ_+ the set of all the positive roots with respect to $\leqq$.

For a $\mathfrak{g}$-module M and $\mu\in\mathfrak{h}^*$, we denote by M_μ the weight space of weight μ. Let $P(M)$ be the set of weights of M. We define a category $\mathcal{O}$ of $\mathfrak{g}$-modules as follows. The objects of $\mathcal{O}$ are $\mathfrak{g}$-modules M satisfying the following conditions.

(i) $M=\sum_{\mu\in\mathfrak{h}^*}M_\mu$ and $\dim M_\mu<+\infty$ for all $\mu\in\mathfrak{h}^*$.

(ii) There exists a finite subset F of $\mathfrak{h}^*$ such that $\mu\leqq\nu$ for some $\nu\in F$ for any $\mu\in P(M)$.

The morphisms of $\mathcal{O}$ are $\mathfrak{g}$-homomorphisms.

We see that all the highest weight modules are objects of $\mathcal{O}$, and so $M(\lambda)$ and $L(\lambda)$ are objects of $\mathcal{O}$ for all $\lambda\in\mathfrak{h}^*$. Any object of $\mathcal{O}$ has a local composition series as follows.

Proposition C [3, Proposition 3.2]. *Let $M\in\mathcal{O}$ and $\lambda\in\mathfrak{h}^*$. There exist a finite sequence $0=M_0\subset M_1\subset\cdots\subset M_t=M$ of $\mathfrak{g}$-submodules of M and a subset J of $\{1, \cdots, t\}$ such that*

$$M_j/M_{j-1}\simeq L(\lambda_j) \quad \text{for some } \lambda_j\geqq\lambda \quad \text{if } j\in J,$$
$$(M_j/M_{j-1})_\mu=0 \quad \text{for any } \mu\geqq\lambda \quad \text{if } j\notin J.$$

We call this sequence a local composition series of M at λ.

Let $M\in\mathcal{O}$ and $\mu\in\mathfrak{h}^*$. Take a $\lambda\in\mathfrak{h}^*$ such that $\mu\geqq\lambda$. Let $0=M_0\subset M_1\subset\cdots\subset M_t=M$ be a local composition series at λ. Then, the multiplicity $[M:L(\mu)]$ is the number of $j\in J$ such that $\lambda_j=\mu$. This number is independent of λ and the local composition series.

§2. Kazhdan-Lusztig conjecture. Let $A=(a_{ij})_{1\leqq i,j\leqq n}$ be the GCM of $\mathfrak{g}$. We take a subset I of the index set $\{1, \cdots, n\}$ of such that $A_I=(a_{ij})_{i,j\in I}$ is the Cartan matrix of a complex semisimple Lie algebra $\mathfrak{g}_I$. We put

$$\mathfrak{h}_I=\sum_{i\in I}\boldsymbol{C}\alpha_i^\vee, \qquad Q_I=\sum_{i\in I}\boldsymbol{Z}\alpha_i,$$
$$\Delta_I=\Delta\cap Q_I, \quad \Delta_{I,+}=\Delta_+\cap Q_I, \quad \mathfrak{n}_{I,\pm}=\sum_{\alpha\in\Delta_{I,+}}\mathfrak{g}_{\pm\alpha}.$$

We see that $\mathfrak{n}_{I,-}+\mathfrak{h}_I+\mathfrak{n}_{I,+}$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{g}_I$. Identify $\mathfrak{g}_I$ with this subalgebra, then, $\mathfrak{h}_I$ is a Cartan subalgebra of $\mathfrak{g}_I$, $\mathfrak{n}_{I,+}$ a nilradical of a Borel subalgebra containing $\mathfrak{h}_I$, and so on.

We define a category $\mathcal{O}_I$ of $\mathfrak{g}_I$-modules by replacing $\mathfrak{g}$ and $\mathfrak{h}$ with $\mathfrak{g}_I$ and $\mathfrak{h}_I$ respectively in the definition of $\mathcal{O}$. Denote by $M_I(\lambda)$ and $L_I(\lambda)$ the Verma module over $\mathfrak{g}_I$ with highest weight $\lambda\in\mathfrak{h}_I^*$ and the unique irreducible quotient of $M_I(\lambda)$, respectively.

We consider the quotients $\mathfrak{h}^*/Q_I$ as additive groups. Let $M\in\mathcal{O}$ and $\Lambda\in\mathfrak{h}^*/Q_I$. Put $M^\Lambda=\sum_{\lambda\in\Lambda}M_\lambda$. Then, $M^\Lambda\in\mathcal{O}_I$ and M decomposes into a direct sum of M^Λ's as a $\mathfrak{g}_I$-module. Moreover, $\mathcal{O}\ni M\mapsto M^\Lambda\in\mathcal{O}_I$ is an exact functor for any $\Lambda\in\mathfrak{h}^*/Q_I$. For each $\lambda\in\mathfrak{h}^*$, we denote by $[\lambda]$ the residue class in $\mathfrak{h}^*/Q_I$ containing λ. Then, we have

Theorem 1. *Let M be a highest weight module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$. Then, $M^{[\lambda]}$ is a highest weight module over $\mathfrak{g}_I$ with highest weight $\lambda|\mathfrak{h}_I$. Moreover we have*

(i) $$M(\lambda)^{[\lambda]} \simeq M_I(\lambda|\mathfrak{h}_I),$$

(ii) $$L(\lambda)^{[\lambda]} \simeq L_I(\lambda|\mathfrak{h}_I).$$

Let $\lambda, \lambda' \in \mathfrak{h}^*$ and $\lambda - \lambda' \in Q_I$. By applying the functor $\mathcal{O} \ni M \mapsto M^{[\lambda]} \in \mathcal{O}_I$ to a local composition series of $M(\lambda)$ at λ' and using Theorem 1, we have the following theorem.

Theorem 2. *Let $\lambda, \lambda' \in \mathfrak{h}^*$ and $\lambda - \lambda' \in Q_I$. Then, the following equality holds*

$$[M(\lambda) : L(\lambda')] = [M_I(\lambda|\mathfrak{h}_I) : L_I(\lambda'|\mathfrak{h}_I)].$$

Let W_I be the Weyl group of $(\mathfrak{g}_I, \mathfrak{h}_I)$. Then, W_I can be canonically identified with a subgroup of W, and the standard order on W_I as a Coxeter group coincides with that induced from W.

Let ρ_I be half the sum of all the elements of $\Delta_{I,+}$. We can prove that $(y\rho - \rho)|\mathfrak{h}_I = y\rho_I - \rho_I$ for all $y \in W_I$, and that the pair $(\lambda, \lambda') = (y\rho - \rho, w\rho - \rho)$ satisfies the condition $\lambda - \lambda' \in Q_I$ for any $y, w \in W_I$. Therefore, if $y, w \in W_I$ and $y \leqq w$, Conjecture B is reduced to Theorem A by Theorem 2. Thus, we get one of our main results as follows.

Theorem 3. *Let $A = (a_{ij})_{1 \leqq i, j \leqq n}$ be a* GCM, *and $\mathfrak{g}$ the Kac-Moody algebra corresponding to A. Let I be a subset of $\{1, \cdots, n\}$ such that $A_I = (a_{ij})_{i,j \in I}$ is the Cartan matrix of a complex semisimple Lie algebra. Under the same notations as above,*

$$[M(y\rho - \rho) : L(w\rho - \rho)] = P_{y,w}(1)$$

holds for all $y, w \in W_I$ such that $y \leqq w$.

Note that the symmetrizability of the GCM A is not assumed here.

§3. Affine Lie algebras. Now, we concentrate on a special case. Let $\mathfrak{g}_0$ be a complex simple Lie algebra, and $\mathfrak{h}_0$ a Cartan subalgebra of $\mathfrak{g}_0$. Denote by Δ_0 the root system of $(\mathfrak{g}_0, \mathfrak{h}_0)$. Let $\boldsymbol{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t with coefficients in $\boldsymbol{C}$. We put $\mathfrak{g} = \boldsymbol{C}d \oplus \boldsymbol{C}c \oplus (\boldsymbol{C}[t, t^{-1}] \otimes_{\boldsymbol{C}} \mathfrak{g}_0)$, and define the bracket in $\mathfrak{g}$ by

$$[c, \mathfrak{g}] = 0, \quad [d, P \otimes x] = t(dP/dt) \otimes x,$$
$$[P \otimes x, P' \otimes x'] = \mathrm{Res}\,((dP/dt)P')K(x, x')c + PP' \otimes [x, x']$$

for all $P, P' \in \boldsymbol{C}[t, t^{-1}]$, $x, x' \in \mathfrak{g}_0$. Here, $\mathrm{Res}\,(P)$ is the coefficient of t^{-1} in P, and $K(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}_0$. $\mathfrak{g}$ is called a *non-twisted* affine Lie algebra, and is one of Kac-Moody algebras. We can (and do) identify $\mathfrak{g}_0$ with the subalgebra $1 \otimes \mathfrak{g}_0$ of $\mathfrak{g}$. In this identification, the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is $\boldsymbol{C}d + \boldsymbol{C}c + \mathfrak{h}_0$. Regard $\mathfrak{h}_0^*$ as a subspace of $\mathfrak{h}^*$ by $\lambda(d) = \lambda(c) = 0$ for $\lambda \in \mathfrak{h}_0^*$, then, $\Delta_0 \subset \Delta$. We denote by $M_0(\lambda)$ the Verma module over $\mathfrak{g}_0$ with highest weight $\lambda \in \mathfrak{h}_0^*$, and $L_0(\lambda)$ the unique irreducible quotient of $M_0(\lambda)$.

We take $\mathfrak{g}_0$ as the subalgebra $\mathfrak{g}_I$ introduced before Theorem 1. Then, the Verma module $M(\lambda)$ $(\lambda \in \mathfrak{h}^*)$ decomposes into a direct sum of $M(\lambda)^{[\lambda - j\delta]}$'s with $j \in \boldsymbol{Z}_{\geqq 0}$ as a $\mathfrak{g}_0$-module, where $\delta \in \mathfrak{h}^*$ is defined by $\delta(d) = 1$, $\delta(c) = 0$, $\delta|\mathfrak{h}_0$

$=0$. So, the following theorem gives a complete branching rule of $M(\lambda)$ as a $\mathfrak{g}_0$-module, together with Theorem 1.

Theorem 4. *Fix a positive integer j. Let $\{\gamma_1, \cdots, \gamma_s\}$ be the set of elements in $\boldsymbol{Z}\Delta_0$ which can be written as a sum of j elements in $\Delta_0 \cup \{0\}$, numbered as $k<l$ if $\gamma_k > \gamma_l$. Then, there exists an increasing sequence*

$$0=M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(s)}=M(\lambda)^{[\lambda-j\delta]}$$

of $\mathfrak{g}_0$-submodules of $M(\lambda)^{[\lambda-j\delta]}$ such that $M^{(k)}/M^{(k-1)}$ is isomorphic to a direct sum of $\mathcal{P}_0(-\gamma_k+j\delta)$-copies of $M_0(\lambda\,|\,\mathfrak{h}_0+\gamma_k)$ for every $k=1, \cdots, s$. Here, for each $\alpha \in Q_+$, we put

$$D=\{(\beta, k) \in (\Delta_+ \setminus \Delta_0) \times \boldsymbol{Z} \,|\, 1 \leqq k \leqq \dim \mathfrak{g}_\beta\},$$

$$\mathcal{P}_0(\alpha)=\sharp\{\tau : D \longmapsto \boldsymbol{Z}_{\geqq 0} \,|\, \alpha=\textstyle\sum_{(\beta,k)\in D} \tau(\beta, k)\beta\}.$$

By this theorem, the problem of computing multiplicities of irreducible subquotients of Verma modules over $\mathfrak{g}$ is reduced to the problem of determining the branching rule of irreducible highest weight modules over $\mathfrak{g}$ as $\mathfrak{g}_0$-modules. Solving this problem, we will get a useful tool to study Conjecture B in full generality for this type of $\mathfrak{g}$.

The author is grateful to Prof. T. Hirai for his useful advice and kind encouragement.

References

[1] A. Beilinson and J. Bernstein: Localisation de $\mathfrak{g}$-modules. C. R. Acad. Sci. Paris, **292**, 15–18 (1981).
[2] J. L. Brylinski and M. Kashiwara: Démonstration de la conjecture de Kazhdan-Lusztig sur les modules de Verma. ibid., **291**, 373–376 (1980).
[3] V. V. Deodhar, O. Gabber and V. G. Kac: Structure of some categories of representations of infinite dimensional Lie algebras. Adv. in Math., **45**, 92–116 (1982).
[4] V. G. Kac: Infinite Dimensional Lie Algebras. Birkhäuser, Boston (1983).
[5] D. A. Kazhdan and G. Lusztig: Representations of Coxeter groups nad Hecke algebras. Invent. Math., **53**, 165–184 (1979).