42. On the Homology Groups of the Mapping Class Groups of Orientable Surfaces with Twisted Coefficients

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1. Introduction. Let Σ_g be a closed orientable surface of genus g and let $\mathcal{M}_g=\pi_0 Diff_+\Sigma_g$ be its mapping class group. Also let $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$ respectively be the mapping class groups of Σ_g relative to the base point $*\in\Sigma_g$ and an embedded disc $D^2\subset\Sigma_g$. It is known that these groups are perfect for all $g\geqq 3$ (see [2, 3]) and Harer determined the second homology group of them in his fundamental paper [2]. The purpose of the present note is to announce our results on the homology groups of them with coefficients in the first homology group $H_1(\Sigma_g, \boldsymbol{Z})$ of Σ_g on which the mapping class groups act naturally.

2. Low dimensional homologies. First we consider the first homology. The results of our previous paper [7] imply

Theorem 1. (i) $H_1(\mathcal{M}_g; H_1(\Sigma_g, \boldsymbol{Z}))\cong \boldsymbol{Z}/2g-2$ $(g\geqq 2)$.

(ii) $H_1(\mathcal{M}_{g,1}; H_1(\Sigma_g, \boldsymbol{Z}))\cong H_1(\mathcal{M}_{g,*}; H_1(\Sigma_g, \boldsymbol{Z}))\cong \boldsymbol{Z}$ $(g\geqq 2)$.

These groups are detected by the crossed homomorphism $f:\mathcal{M}_{g,*}\times H_1(\Sigma_g, \boldsymbol{Z})\to \boldsymbol{Z}$ defined in [7]. Next the second homology group is given by the following Theorem which is one of our main results.

Theorem 2. (i) *$H_2(\mathcal{M}; H_1(\Sigma_g, \boldsymbol{Z}))=0$ for all $g\geqq 12$, where $\mathcal{M}$ stands for any of $\mathcal{M}_g$, $\mathcal{M}_{g,*}$ or $\mathcal{M}_{g,1}$.*

(ii) *$H_2(\mathcal{M}; H_1(\Sigma_g, \boldsymbol{Q}))=0$ for all $g\geqq 9$, where $\mathcal{M}$ is the same as above.*

Corollary 3. $H^2(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Z}))\cong \boldsymbol{Z}/2g-2$ $(g\geqq 9)$.

The group $H^2(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Z}))$ has the following geometric meaning. Choose a generator $o\in H^2(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Z}))$. To any oriented differentiable Σ_g-bundle $\pi: E\to X$, we have associated in [8] a family of Jacobian manifolds $\pi': J'\to X$, which is a *flat* T^{2g}-bundle over X with structure group $H_1(\Sigma_g, \boldsymbol{Z}/2g-2)\rtimes Sp(2g, \boldsymbol{Z})$, and a fibrewise embedding $j': E\to J'$ which induces an isomorphism on the first integral homology on each fibre (topological version of Earle's embedding theorem [1]). We have

Proposition 4 (compare with [1], §8). *Let $\pi: E\to X$ be an oriented Σ_g-bundle. Then the associated family of Jacobian manifolds $\pi': J'\to X$ has a cross-section if and only if $h^*(o)$ vanishes in $H^2(\pi_1(X); H^1(\Sigma_g, \boldsymbol{Z}))$ where $h:\pi_1(X)\to\mathcal{M}_g$ is the holonomy homomorphism of the given Σ_g-bundle and $\pi_1(X)$ acts on $H^1(\Sigma_g, \boldsymbol{Z})$ naturally.*

Corollary 5. *The natural homomorphism $\pi:\mathcal{M}_{g,*}\to\mathcal{M}_g$ induces an isomorphism $H_3(\mathcal{M}_{g,*}, \boldsymbol{Z})\cong H_3(\mathcal{M}_g, \boldsymbol{Z})$ for all $g\geqq 10$. (It is easy to show that the homomorphism $H_3(\mathcal{M}_{g,*}, \boldsymbol{Z})\to H_3(\mathcal{M}_g, \boldsymbol{Z})$ is surjective for all $g\geqq 3$.)*

3. Outline of the proof of Theorem 2. The proof of Theorem 2 is based on Harer's method [2] of computing the second homology group of the mapping class groups which is in turn based on the paper [5] of Hatcher and Thurston. As in [2], let X_2 be the (slightly modified) Hatcher-Thurston complex of the compact surface $\Sigma_g - \mathring{D}^2$ with one boundary component. It is simply connected and the mapping class group $\mathcal{M}_{g,1}$ acts naturally on it cellularly. Harer defines an $\mathcal{M}_{g,1}$-subcomplex $Y_2 \subset X_2$, which is still simply connected and the number of two-cells in its $\mathcal{M}_{g,1}$-orbit is reduced drastically to six. Then he adds two types of three-cells to Y_2 to obtain Y_3 and he uses the standard technique of spectral sequences to deduce his result mentioned above.

We start with Harer's complex Y_3 (with a slight modification of the definition of one of the three-cells because the boundary of his original three-cell is not contained in Y_2). We add five more types of three-cells to Y_3 to obtain Y_3' and then compute the standard spectral sequence which converges to $H_*(Y_3' \times_{\mathcal{M}} K; H_1(\Sigma_g, \boldsymbol{Z}))$ where K is a contractible $\mathcal{M}_{g,1}$-complex. We first construct enough cycles whose homology classes generate $H_2(Y_3' \times_{\mathcal{M}} K; H_1(\Sigma_g, \boldsymbol{Z}))$ and then prove that these cycles are all homologous to zero in $H_2(\mathcal{M}_{g,1}; H_1(\Sigma_g, \boldsymbol{Z}))$. The necessary computations for that are very complicated and lengthy compared with the corresponding ones in the case of constant coefficients. The condition $g \geqq 12$ in the statement of Theorem 2 reflects this situation. Details will be given in [9].

4. Non trivialities of higher homology groups. Harer's stability theorem [3] and Proposition 3-1 of [6] imply

Proposition 6. (i) *The homology group $H_k(\mathcal{M}_g; H_1(\Sigma_g, \boldsymbol{Q}))$ is independent of g in the range $g \geqq 3(k+1)$.*

(ii) *For each prime number p, the homology group $H_k(\mathcal{M}_g; H_1(\Sigma_g, \boldsymbol{Z}/p))$ is independent of g provided $g \geqq 3(k+1)+1$ and p does not divide $2g-2$.*

Remark 7. (i) In the above statements we understand all the homology groups to be abstract vector spaces over $\boldsymbol{Q}$ or $\boldsymbol{Z}/p$. There seems to be no canonical isomorphisms between them. One reason for this is the fact that the Gysin homomorphism (see below) is an *unstable* operation, namely it depends essentially on the genus.

(ii) The statement (i) in the above Proposition does not hold if we replace $H_1(\Sigma_g, \boldsymbol{Q})$ by $H_1(\Sigma_g, \boldsymbol{Z})$ (see Theorem 1, (i)).

Now we consider the cohomology group $H^*(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Q}))$ instead of homology because it is more convenient for the statement of our nontriviality result. As in [6], let $e \in H^2(\mathcal{M}_{g,*}, \boldsymbol{Z})$ be the Euler class of the central extension $0 \to \boldsymbol{Z} \to \mathcal{M}_{g,1} \to \mathcal{M}_{g,*} \to 1$. We define a cohomology class $e_i \in H^{2i}(\mathcal{M}_g, \boldsymbol{Z})$ by setting $e_i = \pi_*(e^{i+1})$ where $\pi_*: H^{2i+2}(\mathcal{M}_{g,*}, \boldsymbol{Z}) \to H^{2i}(\mathcal{M}_g, \boldsymbol{Z})$ is the Gysin homomorphism induced from the projection $\pi: \mathcal{M}_{g,*} \to \mathcal{M}_g$. We call e_i the i-th characteristic class of oriented surface bundles. We also use the same letter e_i for the cohomology class $\pi^*(e_i) \in H^{2i}(\mathcal{M}_{g,*}, \boldsymbol{Z})$.

Making an essential use of Harer's stability theorem [3], we have proved in [6]

Theorem 8. *The homomorphism*

$$\boldsymbol{Q}[e, e_1, e_2, \cdots] \longrightarrow H^*(\mathcal{M}_{g,*}, \boldsymbol{Q})$$

is injective up to degree $(1/3)g$.

Now as was shown in [6] (Proposition 3-1), the Hochschild-Serre spectral sequence $\{E_r^{p,q}, d_r\}$ for the *rational* cohomology group of the extension $1 \to \pi_1(\Sigma_g) \to \mathcal{M}_{g,*} \to \mathcal{M}_g \to 1$ collapses so that we have $E_\infty^{p,q} = E_2^{p,q} = H^p(\mathcal{M}_g; H^q(\Sigma_g, \boldsymbol{Q}))$. Hence if we set

$$K^n(g) = \operatorname{Ker}(\pi_* : H^n(\mathcal{M}_{g,*}, \boldsymbol{Q}) \longrightarrow H^{n-2}(\mathcal{M}_g, \boldsymbol{Q})),$$

then we have a short exact sequence

$$0 \longrightarrow E_\infty^{n,0} = H^n(\mathcal{M}_g, \boldsymbol{Q}) \xrightarrow{\pi^*} K^n(g) \xrightarrow{q} E_\infty^{n-1,1} = H^{n-1}(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Q})) \longrightarrow 0.$$

Now for each natural number i, the cohomology class

$$(2g-2)e^{i+1} + ee_i \in H^{2i+2}(\mathcal{M}_{g,*}, \boldsymbol{Q})$$

is contained in $K^{2i+2}(g)$. Hence we can define an element $v_i \in H^{2i+1}(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Q}))$ by

$$v_i = q((2g-2)e^{i+1} + ee_i).$$

The cup product of v_i with any element of $H^*(\mathcal{M}_g, \boldsymbol{Q})$ belongs to $H^*(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Q}))$ so that we have a homomorphism

$$\boldsymbol{Q}[e_1, e_2, \cdots]\langle v_1, v_2, \cdots\rangle \longrightarrow H^*(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Q})),$$

where the left hand side stands for the free $\boldsymbol{Q}[e_1, e_2, \cdots]$-module with basis $v_1, v_2, \cdots$. With these definitions and notations, we have the following non-triviality result.

Theorem 9. *The homomorphism*

$$\boldsymbol{Q}[e_1, e_2, \cdots]\langle v_1, v_2, \cdots\rangle \longrightarrow H^*(\mathcal{M}_g; H^1(\Sigma_g, \boldsymbol{Q}))$$

is injective up to degree $(1/3)g-1$.

The result of Harer-Zagier [4] implies that the above homomorphism is far from being surjective. However it seems to be reasonable to make the following

Conjecture 10. The homomorphism in Theorem 9 is an isomorphism in the same range.

We can also formulate similar statements to Theorem 9 and Conjecture 10 for the group $\mathcal{M}_{g,*}$, but here we omit them.

Details will appear elsewhere.

References

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