38. An Average Effect of Many Tiny Holes in Nonlinear Boundary Value Problems with Monotone Boundary Conditions

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1. Introduction. Recently H. Attouch has showed in his treatise [1] that it is possible to compute a magnitude effect of many tiny holes, of which shapes are not spherical in general, to the potential term of the Dirichlet problem for the Laplacian, using the notion of capacity. Here we show that the method in [1] can be extended to a different type of boundary conditions, nonlinear boundary conditions, by replacing capacity by a different class of magnitude. This paper is a generalization and an improvement of the work [5].

Let $F(\ni 0)$ be a closed subset of $\boldsymbol{R}^N$, $N\geqq 3$, with non-empty interior and its diameter 2. Let $\boldsymbol{R}^N$ be divided into cubes C_ε^i of volume ε^N and x_ε^i the center of C_ε^i, $i\in \boldsymbol{N}$. We set $F_\varepsilon^i=x_e^i+r_\varepsilon F$ with small $r_\varepsilon>0$. Let Ω be a bounded domain of $\boldsymbol{R}^N$ with smooth boundary Γ. From Ω we remove all holes F_ε^i such that dist $(\Gamma, F_\varepsilon^i)\geqq\varepsilon$ and obtain Ω_ε. We assume that the complement $\boldsymbol{c}F$ of F consists of an unbounded component and has a smooth boundary. We consider the following monotone boundary value problem (cf. [3]): for $f\in L^2(\Omega)$ find $u_\varepsilon\in H^1(\Omega_\varepsilon)$ such that

$$(1)\qquad \begin{cases} -\Delta u_\varepsilon=f & \text{a.e. in } \Omega_\varepsilon,\\ \dfrac{\partial u_\varepsilon}{\partial\nu}+\beta_\varepsilon(u_\varepsilon)=0 & \text{a.e. on } \partial\Omega_\varepsilon,\end{cases}$$

where $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega_\varepsilon$ and β_ε is a function: $\boldsymbol{R}\to\boldsymbol{R}$ defined by (i) $\beta_\varepsilon(r)=(r+c_\varepsilon)/L_\varepsilon$, $r\leqq -c_\varepsilon$, (ii) $\beta_\varepsilon(r)=0$, $|r|\leqq c_\varepsilon$, (iii) $\beta_\varepsilon(r)=(r-c_\varepsilon)/L_\varepsilon$, $r\geqq c_\varepsilon$. The problem (1) admits a unique solution $u_\varepsilon\in H^2(\Omega_\varepsilon)$ (cf. also [3]). We consider the behavior of u_ε under the condition

$$(2)\qquad \sup L_\varepsilon<\infty,\quad c_\varepsilon\to 0,\quad r_\varepsilon\to 0\quad\text{and}\quad n_\varepsilon\to\infty,$$

where n_ε denotes the number of holes of Ω_ε.

We introduce a class of magnitude on a closed set F, determined by the shape of ∂F and a sequence $\{r_\varepsilon, \beta_\varepsilon\}_\varepsilon$ by

$$(3)\qquad C_{\partial F}=\lim_{\substack{R\to\infty\\ r_\varepsilon\to 0}}\gamma(R, r_\varepsilon),$$

$$(4)\qquad \gamma(R, r_\varepsilon)=\inf\left\{\int_{B_R\setminus F}|\nabla v|^2dx+r_\varepsilon\int_{\partial F}v\beta_\varepsilon(v)d\sigma : v\in W_R\right\}$$

where $B_R=\{x\in\boldsymbol{R}^N : |x|<R\}$ and $W_R=\{v\in H^1(B_R\setminus F) : v\geqq 1 \text{ on } \partial B_R\}$. We can show that the value $C_{\partial F}$ is well defined under the condition

$$(5)\qquad L_\varepsilon/r_\varepsilon\to q\qquad\text{as}\quad r_\varepsilon\to 0,$$

where q is a positive constant. In fact, we have

$$C_{\partial F}=(N-2)|\partial B_1|[1+(N-2)q]^{-1}$$

with $F=B_1$, where $|\partial B_1|$ is the $N-1$ dimensional measure. Our result is stated as follows.

Theorem. *Let u_ε be the solution of* (1) *and p a non-negative constant. We assume* (2), (5) *and*

$$n_\varepsilon r_\varepsilon^{N-2}\to p \quad as \quad \varepsilon\to 0. \tag{6}$$

Then we have an extension $\tilde{u}_\varepsilon \in H^1(\Omega)$ of u_ε such that $\tilde{u}_\varepsilon$ converges weakly in $H^1(\Omega)$ to the solution u of

$$\begin{cases}-\Delta u+pC_{\partial F}u|\Omega|^{-1}=f & \text{a.e. } in\ \Omega,\\ u=0 \quad \text{a.e. } on\ \Gamma.\end{cases} \tag{7}$$

2. The well definedness of $C_{\partial F}$. We assume that $r_m\to 0$, $L_m\to 0$ with (5) and $R_m\uparrow\infty$. We denote β_ε, B_R and $\gamma(R,r)$ by β_m, B_m and γ_m with $\varepsilon=\varepsilon_m$, $r=r_m$ and $R=R_m$. Let v_m be the solution of the minimized proplem (4) and let $\bar{v}_m\in H^1_{\mathrm{loc}}(cF)$ the extension of v_m by $\bar{v}_m=1$ on cB_m. We show

$$\liminf_{\substack{n\le m\\ n\to\infty}}(\gamma_n-\gamma_m)\geqq 0. \tag{8}$$

This implies $\inf_n\gamma_n=\lim_n\gamma_n$. For any $\delta>0$ we have n_0 such that $r_n/L_n \leqq 2q^{-1}$, $|r_m/L_m-r_n/L_n|\leqq\delta/(2|\partial F|)$ and $|k_m-k_n|_\infty\leqq q\delta/(4|\partial F|)$, $m\geqq n\geqq n_0$, where $k_n=L_n\beta_n$. By $\bar{v}_n|(B_m\backslash F)\in W_m$ and the definitions of γ_m, v_n and v_m we obtain

$$\gamma_m\leqq\gamma_n+(r_m/L_m-r_n/L_n)\int_{\partial F}v_nk_m(v_n)d\sigma$$
$$+r_nL_n^{-1}\int_{\partial F}v_n(k_m-k_n)(v_n)d\sigma\leqq\gamma_n+\delta.$$

3. Proof of Theorem. We denote u_ε, Ω_ε, $\cup\{F^i_\varepsilon : 1\leqq i\leqq n_\varepsilon\}$ by u_m, Ω_m, F_m, with $\varepsilon=\varepsilon_m$. As stated in [4], we have a uniform bounded family of extensions $E_m : V_m\ni v\to\tilde{v}\in V$, where $V_m=H^1(\Omega_m)$, $V=H^1(\Omega)$. Then we have

$$\int_{\Omega m}(\nabla u_m\nabla v-fv)dx+\int_{\partial\Omega m}v\beta_m(u_m)d\sigma=0, \tag{9}$$

for all $v\in V_m$. Notice $|u_m|\leqq|k_m(u_m)|+c_m$ on $\partial\Omega_m$. Putting $v=u_m$ into (9), using the uniform boundedness of $\{E_m\}_m$, the Poincaré inequality in V, and the inequality $\left||\Omega|^{-1}\int_\Omega v dx\right|\leqq C_\Omega\left[\int_\Omega|\nabla v|dx+\int_\Gamma|v|d\sigma\right]$ for all $v\in W^{1,1}(\Omega)$ with a certain constant C_Ω, we obtain $\sup_m\|\tilde{u}_m\|_V<\infty$ and

$$\sup_m L_m^{-1}\int_{\partial\Omega m}(|w_m|^2+|z_m|^2)d\sigma<\infty, \tag{10}$$

where $w_m=0\vee(\tilde{u}_m-c_m)\in V$, $z_m=0\vee(-\tilde{u}_m-c_m)\in V$. Choose a subsequence still denoted by $\{\tilde{u}_m\}$ such that $\tilde{u}_m\to u$ weakly in V. By (10) we see $u\in H^1_0(\Omega)$. Thus, for the proof it suffices to show that u satisfies

$$\int_\Omega[\nabla u\nabla\zeta+pC_{\partial F}u\zeta|\Omega|^{-1}]dx=\int_\Omega f\zeta dx \qquad \text{for all } \zeta\in C_0^\infty(\Omega). \tag{11}$$

Set $P^i_m=x^i_m+\varepsilon_m2^{-1}B_1$, $Q^i_m=P^i_m\backslash F^i_m$, $P_m=\cup\{P^i_m : 1\leqq i\leqq n_m\}$ and $Q_m=\cup\{Q^i_m : 1\leqq i\leqq n_m\}$. We define $h_m\in W^{1,\infty}(\Omega_m)$ such that (i) $h_m=1$ on $\Omega\backslash P_m$, (ii) $\Delta h_m=0$ on Q_m, (iii) $(\partial h_m/\partial\nu)+\beta_m(h_m)=0$ on ∂F_m. We compare h_m with $\bar{h}_m\in W^{1,\infty}(\Omega_m)$

filling (i), (ii) and (iii′) $\bar{h}_m=0$ on ∂F_m, instead of (iii). We denote by $(\partial/\partial r)$ the outer normal derivative on the boundary ∂P_m of P_m. Then, the measure $T_m=(\partial h_m/\partial r)\delta(\partial P_m)$ belongs to the dual space V^* of V (cf. the proof of theorem 1.27 of [1]), because

(12) $$0\leqq T_m\leqq \bar{T}_m(=(\partial\bar{h}_m/\partial r)\delta(\partial P_m)) \quad\text{and}\quad \bar{T}_m\in V^*.$$

Set $I_m(\zeta)=\int_{\partial F_m}[\beta_m(u_m)h_m-u_m\beta_m(h_m)]\zeta d\sigma$. Putting $v=\zeta h_m$ into (9) we obtain

(13) $$I_m(\zeta)=\int_\Omega [h_m(f\zeta-\nabla\tilde{u}_m\nabla\zeta)+u_m\nabla\zeta\nabla\tilde{h}_m]dx-\langle T_m,\tilde{u}_m\zeta\rangle.$$

In (13) we regard V_m as a subspace of $L^2(\Omega)$ with zero extension. We define the measure U_m by $U_m=|\nabla h_m|^2\delta(Q_m)+h_m\beta_m(h_m)\delta(\partial F_m)$. Modifying the proof of Theorem 1.27 of Attouch [1] and using the definitions of p and $C_{\partial F}$ we get

(14) $$\lim_m \langle U_m, 1\rangle = pC_{\partial F}.$$

Thus, U_m is a positive measure with finite total variation. We have $T_m=U_m+S_m$ over the space $C^\infty(\bar{\Omega})$, where $\langle S_m,\zeta\rangle=\int_{Q_m}h_m\nabla\tilde{h}_m\nabla\zeta dx$, $S_m\in V^*$. By this formula the measure U_m belongs to V^*. By (12) and the strong convergence of $\bar{T}_m$ in V^* (cf. [1]) we see that weak convergence of T_m implies the strong convergence in V^* (cf. lemma 2.8 of [2]). By (14), the uniform boundedness of $\{E_m\}_m$ and the same argument as in [2] we see

(15) $$\tilde{h}_m\longrightarrow 1 \qquad \text{weakly in } V.$$

By (15) and (20) appeared later on we get $h_m\to 1$ in $L^2(\Omega)$. Thus, $S_m\to 0$ weakly in V^*. By (12) $\{U_m\}_m$ is bounded in V^*. By the definitions of $C_{\partial F}$, h_m, we see $\langle U_m,\zeta\rangle\to pC_{\partial F}/|\Omega|\int_\Omega \zeta dx$ for $\zeta\in C^\infty(\bar{\Omega})$ as $m\to\infty$. Thus, by lemma 2.8 of [2] we have

(16) $$T_m\longrightarrow pC_{\partial F}/|\Omega| \qquad \text{strongly in } V^*.$$

Applying the Green theorem to the open set Q_m and the same computation an in (14) we get

(17) $$\lim_m\int_{\partial F_m}\beta_m(h_m)d\sigma=\lim_m\int_{\partial F_m}\frac{\partial h_m}{\partial r}d\sigma=pC_{\partial F}.$$

Let G_m^+, G_m^- and E_m be the characteristic functions of the sets of $\{x\in\partial F_m : w_m>0\}$, $\{x\in\partial F_m : z_m>0\}$ and $\{x\in\partial F_m : h_m-c_m>0\}$, respectively. The definition of β_m implies the following formula.

(18) $$\begin{aligned} I_m(\zeta)=c_m/L_m&\int_{\partial F_m}E_m\zeta[G_m^+(w_m-k_m(h_m))+G_m^-(k_m(h_m)-z_m)]d\sigma\\ &-L_m^{-1}\int_{\partial F_m}E_m\zeta u_m k_m(h_m)(1-G_m^+-G_m^-)d\sigma\\ &+L_m^{-1}\int_{\partial F_m}(1-E_m)\zeta(w_m-z_m)h_m d\sigma.\end{aligned}$$

Thus, by (2), (5), (6), (10), (17) we see that each term of the right hand side of (18) tends to zero. So,

(19) $$I_m(\zeta)\longrightarrow 0 \qquad \text{as} \quad m\longrightarrow\infty.$$

Lemma. *For $\{v_m\in V_m\}_m$ such that $\sup_m\|v_m\|_{L^2(\partial F_m)}<\infty$ we have $\tilde{v}_m-v_m\to 0$ strongly in $L^2(\Omega)$.*

For the proof of Lemma we use the properties of Poisson kernel $P(x, y)$, $x \in F^0$, the interior of F, $y \in \partial F$, such that $0 \leqq P(x, y)$, $\int_{\partial F} P(x, y) d\sigma(y) \leqq 1$ and $\sup \left\{ \int_F P(x, y) dx : y \in \partial F \right\} = M < \infty$. Over these properties we apply the Schwarz inequality, the Fubini theorem and the scaling method as in Example 1 of [6]. By Lemma we get

(20) $\tilde{u}_m - u_m \longrightarrow 0$ and $\tilde{h}_m - h_m \longrightarrow 0$ strongly in $L^2(\Omega)$.

By (13), (15), (16), (19), (20), we see (11). Q.E.D.

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