36. Some Applications of the Generalized Libera Integral Operator

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Summary. The object of the present paper is to prove several interesting characterization theorems involving the generalized Libera integral operator $\mathcal{J}_c$ and a general class $\mathcal{C}(\alpha, \beta)$ of close-to-convex functions in the unit disk. An application of the integral operator $\mathcal{J}_c$ to a class of generalized hypergeometric functions is also considered.

1. Introduction. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z: |z|<1\}$. Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in the unit disk $\mathcal{U}$. Then a function $g(z) \in \mathcal{S}$ is said to be starlike of order α if and only if

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\}>\alpha \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$ and for all $z \in \mathcal{U}$. We denote by $\mathcal{S}^*(\alpha)$ the class of all functions in $\mathcal{S}$ which are starlike of order α. Note that

$$\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^* \subset \mathcal{S} \qquad (0 \leqq \alpha < 1). \tag{1.3}$$

Throughout this paper, it should be understood that functions such as $zg'(z)/g(z)$, which have removable singularities at $z=0$, have had these singularities removed in statements like (1.2).

The class $\mathcal{S}^*(\alpha)$ was introduced by Robertson [8], and was studied subsequently by Schild [9], MacGregor [5], Pinchuk [7], Jack [2], and others.

A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha, \beta)$ if there is a starlike function $g(z)$ of order α such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\beta \tag{1.4}$$

for some β $(0 \leqq \beta < 1)$ and for all $z \in \mathcal{U}$. It follows from (1.4) that

$$\mathcal{C}(\alpha, \beta) \subseteq \mathcal{C}(\alpha, \gamma) \qquad (0 \leqq \gamma \leqq \beta < 1). \tag{1.5}$$

In particular, $\mathcal{C}(0, 0)$ is the familiar class of close-to-convex functions, and $\mathcal{C}(0, \beta)$ is an important subclass of close-to-convex functions. Thus $\mathcal{C}(\alpha, \beta)$ provides an interesting generalization of the class of close-to-convex functions.

In the present paper we make use of the generalized Libera integral operator $\mathcal{J}_c$, defined by Equation (2.1) below, with a view to proving several

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interesting characterization theorems involving the class $\mathcal{C}(\alpha, \beta)$. We also consider an application of the intergral operator $\mathcal{J}_c$ to a class of generalized hypergeometric functions.

2. The generalized Libera integral operator $\mathcal{J}_c$. For a function $f(z)$ belonging to the class $\mathcal{A}$, we define the generalized Libera integral operator $\mathcal{J}_c$ by

$$\mathcal{J}_c(f)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \qquad (c\geqq 0). \tag{2.1}$$

The operator $\mathcal{J}_c$, when $c\in\mathcal{N}=\{1, 2, 3, \cdots\}$, was introduced by Bernardi [1]. In particular, the operator $\mathcal{J}_1$ was studied earlier by Libera [3] and Livingston [4].

The following result will be required in our analysis of the class $\mathcal{C}(\alpha, \beta)$ using the general integral operator $\mathcal{J}_c$:

Lemma (Miller and Mocanu [6, p. 301, Theorem 10]). *Let $M(z)$ and $N(z)$ be regular in the unit disk $\mathcal{U}$ with*

$$M(0)=N(0)=0, \tag{2.2}$$

and let β be real. If $N(z)$ maps $\mathcal{U}$ onto a (possibly many-sheeted) region which is starlike with respect to the origin, then

$$\mathrm{Re}\left\{\frac{M'(z)}{N'(z)}\right\}>\beta(z\in\mathcal{U})\Rightarrow\mathrm{Re}\left\{\frac{M(z)}{N(z)}\right\}>\beta \qquad (z\in\mathcal{U}) \tag{2.3}$$

and

$$\mathrm{Re}\left\{\frac{M'(z)}{N'(z)}\right\}<\beta(z\in\mathcal{U})\Rightarrow\mathrm{Re}\left\{\frac{M(z)}{N(z)}\right\}<\beta \qquad (z\in\mathcal{U}). \tag{2.4}$$

With the help of Lemma, we now prove

Theorem 1. *If the function $f(z)$ defined by* (1.1) *is in the class $\mathcal{C}(\alpha, \beta)$, then*

$$\mathrm{Re}\left\{\frac{z(\mathcal{J}_c(f))'}{\mathcal{J}_c(g)}\right\}>\beta \qquad (z\in\mathcal{U}). \tag{2.5}$$

Proof. A simple computation gives

$$\frac{z(\mathcal{J}_c(f))'}{\mathcal{J}_c(g)}=\frac{z^c f(z)-c\int_0^z t^{c-1}f(t)dt}{\int_0^z t^{c-1}g(t)dt}. \tag{2.6}$$

Setting

$$M(z)=z^c f(z)-c\int_0^z t^{c-1}f(t)dt \tag{2.7}$$

and

$$N(z)=\int_0^z t^{c-1}g(t)dt, \tag{2.8}$$

so that (2.2) is satisfied, we observe that

$$\mathrm{Re}\left\{\frac{M'(z)}{N'(z)}\right\}=\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\beta. \tag{2.9}$$

Thus, by using the lemma, we have

$$\mathrm{Re}\left\{\frac{M(z)}{N(z)}\right\}=\mathrm{Re}\left\{\frac{z(\mathcal{J}_c(f))'}{\mathcal{J}_c(g)}\right\}>\beta, \tag{2.10}$$

which completes the proof of Theorem 1.

Corollary 1. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{S}^*(\beta)$. Then $\mathcal{J}_c(f)$ is also in the class $\mathcal{S}^*(\beta)$.*

Proof. Setting $f(z)=g(z)$ and $\alpha=\beta$ in Theorem 1, we obtain

$$f(z) \in \mathcal{S}^*(\beta) \Rightarrow \operatorname{Re}\left\{\frac{z(\mathcal{J}_c(f))'}{\mathcal{J}_c(f)}\right\} > \beta, \tag{2.11}$$

which proves the corollary.

A natural combination of Theorem 1 and Corollary 1 is contained in

Theorem 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\mathcal{C}(\alpha, \beta)$. Then $\mathcal{J}_c(f)$ defined by* (2.1) *is also in the class $\mathcal{C}(\alpha, \beta)$.*

Proof. Since $g(z) \in \mathcal{S}^*(\alpha)$ for $f(z) \in \mathcal{C}(\alpha, \beta)$, Corollary 1 implies that

$$\mathcal{J}_c(g) \in \mathcal{S}^*(\alpha). \tag{2.12}$$

Applying Theorem 1, we conclude that $f(z)$ satisfies the inequality (2.5) for $g(z) \in \mathcal{S}^*(\alpha)$, that is, that

$$\mathcal{J}_c(f) \in \mathcal{C}(\alpha, \beta), \tag{2.13}$$

which proves Theorem 2.

Putting $\alpha=\beta=0$ in Theorem 2, we have

Corollary 2. *Let the function $f(z)$ defined by* (1.1) *be close-to-convex in the unit disk $\mathcal{U}$. Then $\mathcal{J}_c(f)$ defined by* (2.1) *is also close-to-convex in the unit disk $\mathcal{U}$.*

Remark. Taking $c=1$ in Corollary 2, we obtain the corresponding result given earlier by Libera [3, p. 758, Theorem 3].

Finally, by using the same technique as in proving Theorem 1, we arrive at

Theorem 3. *Let the function $f(z)$ defined by* (1.1) *satisfy the following inequality:*

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} < \beta \qquad (z \in \mathcal{U}) \tag{2.14}$$

for some β $(0 \leqq \beta < 1)$ and $g(z) \in \mathcal{S}^*(\alpha)$. Then

$$\operatorname{Re}\left\{\frac{z(\mathcal{J}_c(f))'}{\mathcal{J}_c(g)}\right\} < \beta \qquad (z \in \mathcal{U}). \tag{2.15}$$

3. Applications to the generalized hypergeometric functions. Let a_j $(j=1, \cdots, p)$ and b_j $(j=1, \cdots, q)$ be complex numbers with

$$b_j \neq 0, \quad -1, \quad -2, \cdots \qquad (j=1, \cdots, q).$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$\begin{aligned} {}_pF_q(z) &\equiv {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!} \qquad (p \leqq q+1), \end{aligned} \tag{3.1}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \begin{cases} 1, & \text{if } n=0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in N. \end{cases} \tag{3.2}$$

We note that the ${}_pF_q$ series in (3.1) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathcal{U}$ if $p=q+1$.

Applying Theorem 2 to the generalized hypergeometric function (3.1), we shall prove

Theorem 4. *Let the function*

$$z\,{}_pF_q(a_1,\cdots,a_p;\,b_1,\cdots,b_q;\,z) \qquad (p\leqq q+1)$$

be in the class $\mathcal{C}(\alpha,\beta)$. *Then the function*

$$z\,{}_{p+1}F_{q+1}(a_1,\cdots,a_p,\,c+1;\,b_1,\cdots,b_q,\,c+2;\,z)$$

is also in the class $\mathcal{C}(\alpha,\beta)$.

Proof. It follows from the definitions (2.1) and (3.1) that

$$\begin{aligned}(3.3)\qquad \mathcal{J}_c(z\,{}_pF_q(z)) &= \frac{c+1}{z^c}\int_0^z t^{c-1}[t\,{}_pF_q(a_1,\cdots,a_p;\,b_1,\cdots,b_q;\,t)]dt\\ &=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{c+1}{n+c+1}\,\frac{z^{n+1}}{n!}\\ &=z\,{}_{p+1}F_{q+1}(a_1,\cdots,a_p,\,c+1;\,b_1,\cdots,b_q,\,c+2;\,z),\end{aligned}$$

which, in view of Theorem 2, yields Theorem 4 immediately.

Finally, by using Theorem 4 repeatedly, we obtain

Corollary 3. *Let the function*

$$z\,{}_pF_q(a_1,\cdots,a_p;\,b_1,\cdots,b_q;\,z)$$

be in the class $\mathcal{C}(\alpha,\beta)$. *Then the function*

$$z\,{}_{p+k}F_{q+k}(a_1,\cdots,a_p,\,c_1+1,\cdots,c_k+1;\,b_1,\cdots,b_q,\,c_1+2,\cdots,c_k+2;\,z)$$

is also in the class $\mathcal{C}(\alpha,\beta)$, *where* $c_j\geqq 0$ $(j=1,\cdots,k)$.

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