35. Entropy of Random Dynamical Systems

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1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let σ be a P-preserving transformation. Given a non-atomic Lebesgue space $(M, \mathcal{B}(M), \mu)$ and a standard measurable space $(S, \mathcal{B}(S))$, consider a $\mathcal{B}(M) \times \mathcal{B}(S) | \mathcal{B}(M)$-measurable map $f: S \times M \ni (s, x) \to f_s x \in M$ and a stationary sequence of S-valued random variables $\{\xi_n\}_{n=1}^{\infty}$ defined by $\xi_n(\omega) = \xi \circ \sigma^{n-1}(\omega)$ for $n \geqq 1$, where ξ is an S-valued random variable. The sequence $X = \{X_n(\omega)\}_{n=0}^{\infty}$ of random maps which are defined by $X_n(\omega) = f_{\xi_n(\omega)} \circ X_{n-1}(\omega)$ $(n \geqq 1)$ and $X_0(\omega) = id_M$, is called a random dynamical system. The purpose of this paper is to define the concept of the (metrical) entropy of such a random dynamical system under the hypothesis that the map $f_s: M \to M$ preserves μ for each $s \in S$.

2. Preliminaries. In what follows, we always identify two subsets of M which coincide with each other up to μ-measure zero. Let α be a countable measurable partition of M and $\mathcal{B}$ be a subσ-algebra of $\mathcal{B}(M)$ (see [3, Ch. 1]).

Put $I(\alpha | \mathcal{B}) = -\sum_{A \in \alpha} \log \mu(A | \mathcal{B})$ where $\mu(A | \mathcal{B})$ denotes the *conditional probability of an event A given* $\mathcal{B}$, and put $H(\alpha | \mathcal{B}) = \int_M I(\alpha | \mathcal{B})(x) \mu(dx)$. They are called the *conditional information of* α *given* $\mathcal{B}$ and the *conditional entropy of* α *given* $\mathcal{B}$ respecively. If $\mathcal{B} = \mathcal{N} = \{\phi, M\}$, $I(\alpha | \mathcal{N}) = -\sum_{A \in \alpha} 1_A \log \mu(A)$ is denoted by $I(\alpha)$ and $H(\alpha | \mathcal{N}) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$ denoted by $H(\alpha)$. They are called the *information of* α and the *entropy of* α respectively. For a countable measurable partition β of M, let $I(\alpha | \beta)$ denote $I(\alpha | \mathcal{B}(\beta))$ and $H(\alpha | \beta)$ denote $H(\alpha | \mathcal{B}(\beta))$ where $\mathcal{B}(\beta)$ is the subσ-algebra of $\mathcal{B}(M)$ generated by the elements of β. Let Z be the set of all countable measurable partition with finite entropy. It is well-known that Z becomes a complete separable metric space with metric ρ defined by $\rho(\alpha, \beta) = H(\alpha | \beta) + H(\beta | \alpha)$ for $\alpha, \beta \in Z$ (see [4]). For $\alpha, \beta \in Z$ and a measurable map $\tau: M \to M$, let $\alpha \vee \beta$ denote the measurable partition $\{A \cap B\,;\, A \in \alpha, B \in \beta\}$ and $\tau^{-1}\alpha$ denote the partition $\{\tau^{-1}A\,;\, A \in \alpha\}$.

3. The main theorems. Unless otherwise stated we use the same notations as before and we assume that f_s preserves the measure μ for each $s \in S$. First, we prove the following:

Theorem 1. *There is a $C(Z)$-valued random variable $h(\alpha, \omega)$ such that*

$$h(\alpha, \omega) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} X_i^{-1}(\omega) \alpha\right) \qquad P\text{-a.e.}$$

and

$$|h(\alpha, \omega)-h(\beta, \omega)|\leqq\rho(\alpha, \beta) \qquad P\text{-a.e.}$$

for any $\alpha, \beta\in Z$, *where* $C(Z)$ *denotes the space of all real valued continuous functions on* Z.

Proof. For fixed $\alpha\in Z$, it is easy to see that

$$H\left(\bigvee_{i=0}^{n+m-1} X_i^{-1}(\omega)\alpha\right)\leqq H\left(\bigvee_{i=0}^{n-1} X_i^{-1}(\omega)\alpha\right)+H\left(\bigvee_{i=0}^{m-1} X_i^{-1}(\sigma^n(\omega))\alpha\right).$$

Therefore the limit $\bar{h}(\alpha, \omega)=\lim_{n\to\infty}(1/n)H(\bigvee_{i=0}^{n-1}X_i^{-1}(\omega)\alpha)$ exists P-a.e. in virtue of the subadditive ergodic theorem (see Theorem 10. 1 in [5, p. 231]). On the other hand, in the same way as Corollary 4.12.1 in [5, p. 91] we can prove that $|\bar{h}(\alpha, \omega)-\bar{h}(\beta, \omega)|\leqq\rho(\alpha, \beta)$ P-a.e. for fixed $\alpha, \beta\in Z$. If we notice that Z is separable, we can take a continuous version $h(\alpha, \omega)$ of $\bar{h}(\alpha, \omega)$. This completes the proof.

This theorem enables us to define the following :

Definition. The (*metrical*) *entropy of the random dynamical system* $X=\{X_n\}_{n=0}^{\infty}$ is the random variable which is given by

$$h(\omega)=\sup_{\alpha\in Z} h(\alpha, \omega).$$

Remark. If the transformation σ is ergodic then $h(\alpha, \omega)$ is constant P-a.e. since it is σ-invariant. In this case we write $h(\alpha)$ and h instead of $h(\alpha, \omega)$ and $h(\omega)$ respectively.

Next we give some properties of the entropy defined above.

Theorem 2 (A Kolmogolov-Sinai type theorem). *Assume that the smallest sub* σ*-algebra which contains all* $\mathcal{B}(\bigvee_{i=0}^{n}X_i^{-1}(\omega)\alpha)$ *coincides with* $\mathcal{B}(M)$ P-a.e. *for some* $\alpha\in Z$. *Then we have* $h(\omega)=h(\alpha, \omega)$ P-a.e.

Proof. For any positive integer m and any $\beta\in Z$, we have

$$\begin{aligned} H\left(\bigvee_{i=0}^{k-1} X_i^{-1}(\omega)\beta\right) &\leqq H\left(\bigvee_{i=0}^{k-1} X_i^{-1}(\omega)\bigvee_{j=0}^{m-1} X_j^{-1}(\sigma^i\omega)\alpha\right)\\ &\quad+H\left(\bigvee_{i=0}^{k-1} X_i^{-1}(\omega)\beta\,\middle|\,\bigvee_{i=0}^{k-1} X_i^{-1}(\omega)\bigvee_{j=0}^{m-1} X_j^{-1}(\sigma^i\omega)\alpha\right)\\ &\leqq H\left(\bigvee_{i=0}^{m+k-2} X_i^{-1}(\omega)\alpha\right)+\sum_{i=0}^{k-1} H\left(\beta\,\middle|\,\bigvee_{j=0}^{m-1} X_j^{-1}(\sigma^j\omega)\alpha\right). \end{aligned}$$

Here the first inequality follows from the fact that $H(\beta_1\vee\beta_2)=H(\beta_1)+H(\beta_2|\beta_1)$ and the second inequality follows from the fact that

$$\bigvee_{i=0}^{m+k-2} X_i^{-1}(\omega)\alpha=\bigvee_{i=0}^{k-1} X_i^{-1}(\omega)\bigvee_{j=0}^{m-1} X_j^{-1}(\sigma^i\omega)\alpha$$

and $H(\bigvee_{i=1}^{n}\alpha_i|\bigvee_{i=1}^{n}\beta_i)\leqq\sum_{i=1}^{n}H(\alpha_i|\beta_i)$ for any $\{\alpha_i\}_{i=1}^{n}, \{\beta_i\}_{i=1}^{n}\subset Z$. Putting $f_m(\omega)=H(\beta|\bigvee_{j=0}^{m-1}X_j^{-1}(\omega)\alpha)$, we have

$$H\left(\bigvee_{i=0}^{k-1} X_i^{-1}(\omega)\beta\right)\leqq H\left(\bigvee_{i=0}^{m+k-2} X_i^{-1}(\omega)\alpha\right)+\sum_{i=0}^{k-1} f_m(\sigma^i\omega).$$

Therefore in virtue of the ergodic theorem, we have

$$h(\beta, \omega)\leqq h(\alpha, \omega)+\bar{f}_m(\omega) \qquad P\text{-a.e.},$$

where $\bar{f}_m(\omega)=\lim_{n\to\infty}(1/n)\sum_{j=0}^{n-1}f_m(\sigma^j\omega)$. From the assumption, we can show that $f_m\to 0$ $(m\to\infty)$ P-a.e. Thus we have $\boldsymbol{E}\bar{f}_m=\boldsymbol{E}f_m\to 0$ $(m\to\infty)$. Since $\bar{f}_m\geqq 0$, we may assume that $\bar{f}_m\to 0$ P-a.e. $(m\to\infty)$. This implies that $h(\alpha, \omega)=h(\omega)$ P-a.e.

For a random dynamical system we introduce a transformation $T: M \times \Omega \to M \times \Omega$ defined by

$$T(x, \omega) = (X_1(\omega)x, \sigma\omega) \qquad \text{for } (x, \omega) \in M \times \Omega.$$

It is easy to see that the product measure $\mu \times P$ is T-invariant. For $\alpha \in Z$, put $f(x, \omega) = \lim_{n\to\infty} I(\alpha \mid \bigvee_{i=1}^{n} X_i^{-1}(\omega)\alpha)(x)$ if the limit exists, $=\infty$ otherwise and $f_\alpha(x, \omega) = \lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} f \circ T^i(x, \omega)$ if the limit exists, $=\infty$ otherwise. In particular, these limits exist $\mu \times P$-a.e. and in $L^1(\mu \times P)$ in virtue of Doob's theorem and the ergodic theorem. Then we have the following random version of the Shannon-McMillan Theorem.

Theorem 3. $(1/n) I(\bigvee_{i=0}^{n} X_i^{-1}(\omega)\alpha) \to f_\alpha(x, \omega)$ $\mu \times P$-a.e. *and in* $L^1(\mu \times P)$ *as* $n \to \infty$.

Corollary. *If the transformation T is ergodic then* $-(1/n)\log \mu(A_n(x, \omega)) \to h(\alpha)$ *P-a.e., where* $A_n(x, \omega)$ *is the element of* $\bigvee_{i=0}^{n-1} X_i^{-1}(\omega)\alpha$ *which contains* $x \in M$.

Remarks. 1) Since the measure theoretical dynamical system (σ, P) is a factor of $(T, \mu \times P)$, σ is ergodic if T is ergodic (see [4]). This is the reason why we use the notation $h(\alpha)$ in the Corollary.

2) Consider the case $\{\xi_n\}_{n=1}^{\infty}$ are mutually independent and the σ-algebra $\mathcal{B}$ is generated by them. Then, T is ergodic if the measure theoretical dynamical system $(f_{\xi(\omega)}, \mu)$ is ergodic with positive P-measure.

Proof of Theorem 3. It is not hard to see that

$$I\left(\bigvee_{i=0}^{n-1} X_i^{-1}(\omega)\alpha\right)(x) = I\left(\alpha \,\middle|\, \bigvee_{i=1}^{n-1} X_i^{-1}(\omega)\alpha\right)(x) + \cdots$$
$$+ I\left(\alpha \,\middle|\, \bigvee_{i=1}^{n-2} X_i^{-1}(\sigma\omega)\alpha\right)(X_1(\omega)x) + \cdots + I(\alpha)(X_{n-1}(\omega)x).$$

Put $f_k(x, \omega) = I(\alpha \mid \bigvee_{i=1}^{k-1} X_i^{-1}(\omega)\alpha)(x)$. Clearly, we have

$$\left|\frac{1}{n} I\left(\bigvee_{i=0}^{n-1} X_i^{-1}(\omega)\right) - f_\alpha(x, \omega)\right| \leq \frac{1}{n}\sum_{i=0}^{n-1} |f_{n-1} \circ T^i(x, \omega) - f(x, \omega)|$$
$$+ \left|\frac{1}{n}\sum_{i=0}^{n-1} f \circ T^i(x, \omega) - f_\alpha(x, \omega)\right|.$$

The last term goes to 0 $\mu \times P$-a.e. and in $L^1(\mu \times P)$ as $n \to \infty$. We must prove that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} g_{n-i} \circ T^i(x, \omega) = 0$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int g_{n-i} \circ T^i(x, \omega)\mu(dx)P(d\omega) = 0,$$

where $g_k = f_k - f$. But this can be done in the same way as the proof of Theorem 2.5 in [3, p. 21].

4. Other results. If M has a topologically rich structure we can obtain the following:

Theorem 4 (A random version of Katok's theorem [2]). *We assume that M is a compact metric space with metric d and* f_s *is a continuous map on M for each* $s \in S$. *We further assume that the transformation T, which* *is introduced in the previous section, is ergodic. Then we have, for* $\delta>0$,

$$h=\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon,\delta,\omega)=\lim_{\varepsilon\downarrow 0}\liminf_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon,\delta,\omega).$$

Here $N(n,\varepsilon,\delta,\omega)$ *stands for the minimal number of* ε*-balls in the* $d_{n,\omega}$*-metric which cover the set of* μ*-measure more than or equal to* $1-\delta$, *where* $d_{n,\omega}$*-metric is defined by*

$$d_{n,\omega}(x,y)=\max_{0\leq i\leq n-1} d(X_i(\omega)x, X_i(\omega)y) \qquad \textit{for } x, y\in M.$$

Theorem 5 (A random version of Kushinirenko's theorem). *Assume that* M *is a compact smooth manifold without boundary and* f_s *is a* C^1*-differentiable map on* M *for each* $s\in S$. *If*

$$E\log^+\|f_{\xi(\cdot)}\|_{C^1}=\int\log^+\|f_{\xi(\omega)}\|_{C^1}P(d\omega)<\infty,$$

then we have $h(\omega)<\infty$ P-a.e., *where* $\|\cdot\|_{C^1}$ *denotes the* C^1*-norm of a* C^1*-differential map* $\cdot$.

The proofs of Theorem 4, and Theorem 5 are not difficult but complicated and quite long because we must modify the proofs of deterministic cases. For example, for the proof of Theorem 4, we need a modification of the proof of Theorem 1. 1 in [2] and for the proof of Theorem 5, we need modifications of the proof of Corollary to Lemma 18.2 in [1] and the proof of Theorem 7.5 in [5, p. 181]. Detailed proofs of theorems in this paper will be given elsewhere.

References

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