25. The L^p-boundedness of Pseudo-differential Operators Satisfying Estimates of Parabolic Type and Product Type. II

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We stated in our previous paper (Yamazaki [6]) the L^p-boundedness of pseudo-differential operators with non-smooth symbols satisfying non-classical estimates. A proof will be given in the forthcoming paper (Yamazaki [7]).

On the other hand, Bourdaud [1] and Nagase [4] generalized the L^p-boundedness theorem of Coifman-Meyer [2] and Muramatu-Nagase [3] on the classical symbols, by considering the combined effect of the x-regularity and the ξ-growth of the symbols.

Here we consider a similar effect where the symbols satisfy non-classical estimates. Our main theorem is an improvement of Theorem 4 of [7].

1. Notations and definitions. Let $n(1), \cdots, n(N)$ be positive integers. We put $n=n(1)+\cdots+n(N)$ and

$$\Lambda(\nu)=\{l \in N;\ n(1)+\cdots+n(\nu-1)+1 \leqq l \leqq n(1)+\cdots+n(\nu)\}$$

for $\nu=1, \cdots, n$.

We regard R^n as $R^{n(1)}\times\cdots\times R^{n(N)}$, and write $x \in N^n$ as $x=(x^{(1)}, \cdots, x^{(N)})$, where $x^{(\nu)}=(x_l)_{l\in\Lambda(\nu)}$. We also give a weight $M=(M^{(1)}, \cdots, M^{(N)})$ to the coordinate variables of R^n, where each $M^{(\nu)}=(m_l)_{l\in\Lambda(\nu)}$ satisfies the condition $\min_{l\in\Lambda(\nu)} m_l=1$.

Next, for every $\nu=1, \cdots, N$, we define a function $[y]_\nu$ of $y=(y_l)_{l\in\Lambda(\nu)} \in R^{n(\nu)}$ with values in $R^+=\{t;\ t\geqq 0\}$ as follows. We put $[0]_\nu=0$, and if $y\neq 0$, let $[y]_\nu$ denote the unique positive root of the equation $\sum_{l\in\Lambda(\nu)} t^{-2m_l}y_l^2=1$ with respect to t.

Further, for $\nu=1, 2, \cdots, N$ and $y \in R^{n(\nu)}$, let $\Delta_y^{(\nu)}$ denote the difference of the first order with respect to the ν-th part of the coordinate variables; that is, we put

$$\Delta_y^{(\nu)} f(x)=f(x^{(1)}, \cdots, x^{(\nu)}-y, \cdots, x^{(N)})-f(x)$$

for a function $f(x)$ on R^n. We also fix a positive number L.

Now we introduce a notion to state our main theorem.

Definition. We call a family of functions $\{\omega_1(s_1;t_1), \omega_2(s_1, s_2;t_1, t_2), \cdots, \omega_N(s_1, s_2, \cdots, s_N;t_1, t_2, \cdots, t_N)\}$ a *multiple modulus of growth and continuity* if it satisfies the following four conditions:

1) For every ν, the function $\omega_\nu(s_1, \cdots, s_\nu;t_1, \cdots, t_\nu)$ is a function on $(R^+)^{2\nu}$ into R^+, and is monotone-increasing and concave with respect to each t_k.

2) There exists a constant C such that the inequality

$$\omega_\nu(s_1', s_2, \cdots, s_\nu; t_1, \cdots, t_\nu) \leqq C\omega_\nu(s_1, s_2, \cdots, s_\nu; t_1, \cdots, t_\nu)$$

holds for $s_1/2 \leqq s_1' \leqq 2s_1$.

3) $\omega_\nu(s_1, \cdots, s_\nu; t_1, \cdots, t_\nu)$ is invariant under any permutation of the ordered pairs $(s_1, t_1), (s_2, t_2), \cdots, (s_\nu, t_\nu)$.

4) For each $1 \leqq \mu < \nu \leqq N$, we have

$$\omega_\nu(s_1, \cdots, s_\nu; t_1, \cdots, t_\nu) \leqq 2^{L(\nu-\mu)}\omega_\mu(s_1, \cdots, s_\mu; t_1, \cdots, t_\mu).$$

Then, given a multiple modulus of growth and continuity $\{\omega_1(s_1; t_1), \cdots, \omega_N(s_1, \cdots, s_N; t_1, \cdots, t_N)\}$, consider the conditions $(*\mu)$ for $\mu=0, 1, \cdots, N$ as follows:

$(*0)$ For every $\nu=1, 2, \cdots, N$, $l \in \Lambda(\nu)$ and $k=0, 1, \cdots, n+1$, we have $|\partial_{\xi_l}^k P(x, \xi)| \leqq C(1+[\xi^{(\nu)}]_\nu)^{-m_l k}$.

$(*\mu)$ $(\mu=1, \cdots, N)$ For every $\nu=1, 2, \cdots, N$, $l \in \Lambda(\nu)$, $1 \leqq \nu(1) < \nu(2) < \cdots < \nu(\mu) \leqq N$, $y_1 \in \boldsymbol{R}^{n(\nu(1))}, \cdots, y_\mu \in \boldsymbol{R}^{n(\nu(\mu))}$, $l \in \Lambda(\nu)$ and $k=0, 1, \cdots, n+1$, we have

$$(1) \qquad |(\Delta_{y_1}^{(\nu(1))})^L(\cdots((\Delta_{y_\mu}^{(\nu(\mu))})^L(\partial_{\xi_l}^k P(x, \xi)))\cdots)| \leqq C\omega_\mu(1+[\xi^{(\nu(1))}]_{\nu(1)}, \cdots, 1+[\xi^{(\nu(\mu))}]_{\nu(\mu)}; [y_1]_{\nu(1)}, \cdots, [y_\mu]_{\nu(\mu)}) \times (1+[\xi^{(\nu)}]_\nu)^{-m_l k}.$$

2. Statement of the theorem and remarks. Our main result is the following theorem.

Theorem. *The following three conditions concerning moduli of growth and continuity are equivalent:*

1) $$\int_0^1 \cdots \int_0^1 \omega_\nu\left(\frac{1}{t_1}, \cdots, \frac{1}{t_\nu}; t_1, \cdots, t_\nu\right)^2 \cdot \frac{dt_1 \cdots dt_\nu}{t_1 \cdots t_\nu} < \infty$$

holds for every $\nu=1, \cdots, N$.

2) *If a symbol* $P(x, \xi)$ *satisfies the conditions* $(*\mu)$ *for all* $\mu=0, 1, \cdots, N$, *then the associated pseudo-differential operator* $P(x, D)$ *defined by the formula*

$$P(x, D)u(x) = (2\pi)^{-n}\int_{\boldsymbol{R}^n} \exp(ix \cdot \xi) P(x, \xi)\hat{u}(\xi) d\xi$$

is bounded on $L^p(\boldsymbol{R}^n)$ *for every* $1 < p < \infty$.

3) *For every symbol* $P(x, \xi)$ *satisfying the conditions* $(*\mu)$ *for all* $\mu=0, 1, \cdots, N$, *there exists* $1 < p < \infty$ *such that the associated operator* $P(x, D)$ *is bounded on* $L^p(\boldsymbol{R}^n)$.

Remark 1. If $\{\omega_1(t_1), \cdots, \omega_N(t_1, \cdots, t_N)\}$ is a modulus of continuity in the sense of Yamazaki [6], [7], and if $\Omega_1(s_1), \cdots, \Omega_N(s_1, \cdots, s_N)$ are functions satisfying the inequalities such as $\Omega_1(s_1') \leqq C\Omega_1(s_1)$ for $s_1/2 \leqq s_1' \leqq 2s_1$, then $\{2^L\Omega_1(s_1)\omega_1(t_1), \cdots, 2^{LN}\Omega_N(s_1, \cdots, s_N)\omega_N(t_1, \cdots, t_N)\}$ is a multiple modulus of growth and continuity.

In case $N=L=1$ and $M^{(1)}=(1, \cdots, 1)$, the main theorem with this type of multiple modulus of growth and continuity coincides with Theorem 2 of Bourdaud [1]. Yabuta [5] also considered symbols satisfying estimates of this type, and obtained boundedness properties on more general function spaces.

Remark 2. As a special case in the above remark, consider the case

$\Omega_\nu \equiv 1$ for all ν. Then we obtain Theorem 3 of Yamazaki [7].

Remark 3. Theorem 4 of [7] asserts the L^p-boundedness of the operators with symbols satisfying the condition $(*\mu)$ with (1) replaced by

$$|(\Delta_{y_1}^{(\nu(1))})^L(\cdots((\Delta_{y_\mu}^{(\nu(\mu))})^L(\partial_{\xi_l}^k P(x,\xi)))\cdots)|$$
$$\leqq C\omega_\mu(|y_1|, \cdots, |y_\mu|)\Omega([\xi^{(1)}]_1)\times\cdots\times\Omega([\xi^{(N)}]_N)(1+[\xi^{(\nu)}]_\nu)^{-m_l k}.$$

Remark 4. Let $\{\omega_1, \cdots, \omega_N\}$ be the same as in Remark 1, and let δ be a constant such that $0\leqq\delta<1$. Suppose that $P(x,\xi)$ satisfies the condition $(*\mu)$ with the estimate (1) replaced by

$$(2)\qquad |(\Delta_{y_1}^{(\nu(1))})^L(\cdots((\Delta_{y_\mu}^{(\nu(\mu))})^L(\partial_{\xi_l}^k P(x,\xi)))\cdots)|$$
$$\leqq C\omega_\mu([y_1]_{\nu(1)}\cdot(1+[\xi^{(\nu(1))}]_{\nu(1)})^\delta, \cdots,$$
$$[y_\mu]_{\nu(\mu)}(1+[\xi^{(\nu(\mu))}]_{\nu(\mu)})^\delta)\cdot(1+[\xi^{(\nu)}]_\nu)^{-m_l k}.$$

Then it follows that $P(x,\xi)$ satisfies the condition $(*\mu)$ with the multiple modulus of growth and continuity $\{\omega_1(s_1^\delta t_1), \cdots, \omega_N(s_1^\delta t_1, \cdots, s_N^\delta t_N)\}$. But this satisfies the condition 1) of the main theorem if and only if

$$\int_0^1\cdots\int_0^1 t_1^{-1}\cdots t_\nu^{-1}\omega_\nu(t_1, \cdots, t_\nu)^2 dt_1\cdots dt_\nu<\infty$$

holds for every ν. Thus we obtain a result on the L^p-boundedness of the operators with symbols satisfying the estimates (2).

In case $N=L=1$ and $M^{(1)}=(1, \cdots, 1)$, Nagase [4] considered the symbols satisfying similar estimates as above and weaker differentiability conditions with respect to ξ, and obtained the L^p-boundedness for $2\leqq p<\infty$.

Remark 5. If $\omega_1(s;t)$ satisfies $\omega_1(t^{-1};t)\leqq\omega_1(t'^{-1};t')$ for $t\leqq t'$ and

$$\int_0^1 t^{-1}(-\log t)^{N-1}\omega_1(t^{-1};t)^2 dt<\infty,$$

any multiple modulus of growth and continuity satisfies the condition 1) of the main theorem automatically. (See [6] or [7].)

3. Outline of the proof. It is clear that the operator constructed in the same way as in Section 4 of [7] implies the necessity of the condition 1). Hence we have only to show the sufficiency. We employ the same notations as in [7].

For a symbol $P(x,\xi)$ satisfying the conditions $(1.\mu)$ for $\mu=0, 1, \cdots, N$, let $a_{K,h}(x)$ and $a_{K,h,A}(x)$ be the same as in Section 5 of [7]. Then, in the same way as we have obtained the estimate (5.7) of [7], we obtain

$$|(\Delta_{y_1}^{(1)})^L(\cdots((\Delta_{y_\nu}^{(\nu)})^L a_{K,h}(x))\cdots)|$$
$$\leqq C\omega_\nu(2^{k_1}, \cdots, 2^{k_\nu}; [y_1]_1, \cdots, [y_\nu]_\nu)\ (1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}.$$

Suppose that $a(j)=0$ for $j\leqq\nu$ and that $a(j)>0$ for $j>\nu$. Then, in view of the monotonicity and the concavity of ω_ν, we obtain

$$|a_{K,h,A}(x)|\leqq C\int\omega_\nu(2^{k_1}, \cdots, 2^{k_\nu}; [2^{-k_1M^{(1)}}y_1]_1, \cdots, [2^{-k_\nu M^{(\nu)}}y_\nu]_\nu)$$
$$\times\prod_{j=1}^N|F^{-1}[\Psi(4[\xi^{(j)}]_j)](y_j)|\,dy\cdot(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}$$
$$\leqq C(1+|h_1|^{n+1}+\cdots+|h_n|^{n+1})^{-1}\omega_\nu(2^{k_1}, \cdots, 2^{k_\nu}; 2^{-k_1}, \cdots, 2^{-k_\nu})$$

in the same way as in Section 5 of [7]. Hence it suffices to show

$$\sum_{K\in N^\nu}\omega_\nu(2^{k_1}, \cdots, 2^{k_\nu}; 2^{-k_1}, \cdots, 2^{-k_\nu})^2<\infty.$$

But this follows from the condition 1) of the main theorem.

This completes the proof.

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