16. A Remark on Ergodic Theorems for Pseudo-Resolvents

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In the article [3] K. Yosida has shown the ergodic theorems for pseudo-resolvents. In this note we give another formulation and a fairly short proof of these theorems.

1. Notations. By X we denote a Hausdorff topological linear space, and by $\mathcal{L}(X)$ we denote the algebra of all continuous linear operators from X into X. For a linear operator T by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ we denote the domain, the range and the null space of T, respectively. For a subset M of X by M^a we denote its closure, and a sequence $\{\lambda_n\}$ of complex numbers is said to be a zero sequence if $\lambda_n \to 0$ as $n \to \infty$.

2. A pseudo-resolvent $\{R_\lambda\}_{\lambda \in D}$ is an $\mathcal{L}(X)$-valued function on a subset D of complex plane satisfying the resolvent equation

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \qquad (\lambda, \mu \in D), \tag{1}$$

so that $R_\lambda R_\mu = R_\mu R_\lambda$.

Let $\{R_\lambda\}_{\lambda \in D}$ be a pseudo-resolvent, and set

$$A_\lambda = I - \lambda R_\lambda \qquad \text{for } \lambda \in \Delta = D, \tag{2}$$

where I stands for the identity operator, or

$$A_\lambda = \lambda^{-1} R_{\lambda^{-1}} \qquad \text{for } \lambda \in \Delta = \{\lambda ; \lambda^{-1} \in D\}. \tag{3}$$

Then it is easy to see that the identity

$$\lambda A_\lambda = \{\mu I + (\lambda - \mu) A_\lambda\} A_\mu \tag{4}$$

holds for any $\lambda, \mu \in \Delta$. Therefore all A_λ, $\lambda \in \Delta$, have a common range and a common null space.

Our formulation of ergodic theorems are stated as follows:

Theorem 1. *Let $\{R_\lambda\}$ be a pseudo-resolvent on D and assume that the family $\{\lambda R_\lambda\}_{\lambda \in D}$ of operators is equicontinuous.*

In the case where 0 is an accumulation point of D we define A_λ by (2), and in the case where ∞ is an accumulation point of D we define A_λ by (3). In each case let R and N be the common range and the common null space of A_λ, $\lambda \in \Delta$, respectively.

Then;

(a) $R^a \cap N = \{0\}$.

(b) *The following four conditions are equivalent;*

(i) $x \in R^a + N$,

(ii) *$\{\psi(A_\lambda)x\}$ converges as $\lambda \to 0$ for any polynomial ψ,*

(iii) *There exist a polynomial ψ with $\psi(0) \neq \psi(1)$ and a zero sequence $\{\lambda_n\}$ in Δ such that $\{\psi(A_{\lambda_n})x\}$ converges as $n \to \infty$,*

(iv) *There exist a polynomial ψ with $\psi(0) \neq \psi(1)$, a zero sequence $\{\lambda_n\}$*

in Δ and a point y in X such that $x=\lim_{n\to\infty}\psi(A_{\lambda_n})y$.

(c) *The following four conditions are equivalent;*

(i) $x\in R^a$,

(ii) $\psi(A_\lambda)x\to\psi(1)x$ *as* $\lambda\to 0$ *for any polynomial* ψ,

(iii) *There exist a polynomial* ψ *with* $\psi(0)\neq\psi(1)$ *and a zero sequence* $\{\lambda_n\}$ *in* Δ *such that* $\lim_{n\to\infty}\psi(A_{\lambda_n})x=\psi(1)x$,

(iv) *There exist a polynomial* ψ, *a zero sequence* $\{\lambda_n\}$ *in* Δ *and a point* y *in* X *such that* $x=\lim_{n\to\infty}\psi(A_{\lambda_n})y-\psi(0)y$.

(d) *The following four conditions are equivalent;*

(i) $x\in N$,

(ii) $\psi(A_\lambda)x=\psi(0)x$ *for any polynomial* ψ *and any* λ *in* Δ,

(iii) *There exist a polynomial* ψ *with* $\psi(0)\neq\psi(1)$ *and a zero sequence* $\{\lambda_n\}$ *in* Δ *such that* $\lim_{n\to\infty}\psi(A_{\lambda_n})x=\psi(0)x$,

(iv) *There exist a polynomial* ψ, *a zero sequence* $\{\lambda_n\}$ *in* Δ *and a point* y *in* X *such that* $x=\lim_{n\to\infty}\psi(A_{\lambda_n})y-\psi(1)y$.

3. Before the proof of the theorem we notice the following lemmas, which are easily seen.

Lemma 1. *Let a family* $\{T_\lambda\}$ *of linear operators on* X *be equicontinuous. Then the family* $\{\psi(T_\lambda)\}$ *is also equicontinuous for any polynomial* ψ.

Lemma 2. *Let a family* $\{T_\lambda\}$ *of linear operators on* X *be equicontinuous. If* $x_\lambda\to 0$ *as* $\lambda\to\lambda_0$, *then* $T_\lambda x_\lambda\to 0$ *as* $\lambda\to\lambda_0$.

Lemma 3. *Let a family* $\{T_\lambda\}$ *of linear operators on* X *be equicontinuous. Then the subspace* $\{x\,;\,\lim_{\lambda\to\lambda_0}T_\lambda x=0\}$ *is closed.*

4. Proof of Theorem 1 part (c). Let $x=A_\mu y$ for some μ in Δ and y in x, and let ψ be a polynomial. Since $\psi(t)-\psi(1)=\phi(t)(t-1)$, where ϕ is a polynomial, and since $\{\phi(A_\lambda)(A_\lambda-A_\mu)\}$ is equicontinuous by Lemma 1, it follows from (4) and Lemma 2 that

$$\begin{aligned}\psi(A_\lambda)x-\psi(1)x&=\phi(A_\lambda)(A_\lambda-I)A_\mu y\\&=\phi(A_\lambda)(A_\lambda-A_\mu)\lambda(\lambda-\mu)^{-1}y\to 0\end{aligned}$$

as $\lambda\to 0$. Thus we have

$$R\subset P_\psi=\{x\,;\,\lim_{\lambda\to 0}\{\psi(A_\lambda)-\psi(1)\}x=0\}.$$

By Lemma 3 P_ψ is closed, so that it also contains R^a. Therefore (i) implies (ii).

Obviously, (iii) follows from (ii), and (iv) follows from (iii) by setting $y=\{\psi(1)-\psi(0)\}^{-1}x$. Finally, making use of the formula $\psi(t)=\psi(0)+t\phi_0(t)$, where ϕ_0 is a polynomial, we see that if (iv) is satisfied then

$$x=\lim_{n\to\infty}\{\psi(A_{\lambda_n})-\psi(0)\}y=\lim_{n\to\infty}A_{\lambda_n}\phi_0(A_{\lambda_n})y\in R^a.$$

5. Proof of Theorem 1 part (d). It is obvious that (i) implies (ii) and that (ii) implies (iii). The condition (iv) follows from (iii) by setting $y=\{\psi(0)-\psi(1)\}^{-1}x$. Assume that (iv) is satisfied. Then by part (c) (ii) in Theorem 1 we have

$$A_\mu x=\lim_{n\to\infty}\psi(A_{\lambda_n})A_\mu y-\psi(1)A_\mu y=\psi(1)A_\mu y-\psi(1)A_\mu y=0$$

for any μ in Δ, which implies $x\in N$.

6. Proof of Theorem 1 part (a). Let $x \in R^a \cap N$. Then by (ii) in (c) $A_\lambda x = 0 \to 1 \cdot x$ as $\lambda \to 0$, which implies $x=0$.

7. Proof of Theorem 1 part (b). Let $x = x_1 + x_2$, $x_1 \in R^a$, $x_2 \in N$. Then, by (ii) in (c) and (d), we have

$$\psi(A_\lambda)x = \psi(A_\lambda)x_1 + \psi(0)x_2 \to \psi(1)x_1 + \psi(0)x_2 \quad \text{as } \lambda \to 0,$$
$$(A_\lambda + I)(2^{-1}x_1 + x_2) \to x = x_1 + x_2 \quad \text{as } \lambda \to 0.$$

Therefore (i) implies (ii) and (iv).

Obviously, (ii) implies (iii). Assume that (iii) is satisfied, and let w be the limit of a sequence $\{\psi(A_{\lambda_n})x\}$, where $\psi(0) \neq \psi(1)$. Then by part (c) and (d) of the theorem we have

$$w - \psi(0)x \in R^a, \qquad w - \psi(1)x \in N,$$

which implies (i), because of $\psi(0) \neq \psi(1)$. Finally, assume that (iv) is satisfied. Then by (c) and (d) we have

$$x - \psi(0)y \in R^a, \qquad x - \psi(1)y \in N,$$

so that by $\psi(0) \neq \psi(1)$ x belongs to $R^a + N$. This completes the proof of Theorem 1.

8. Corollary. *Under the same assumptions as in Theorem* 1 *we have*

(5) $$R^a = \mathcal{R}(A_\lambda^m)^a, \qquad N = \mathcal{N}(A_\lambda^m),$$

for any $\lambda \in \Delta$ *and any positive integer* m.

Proof. Set $\psi(t) = t^m$. If $x \in R^a$, then $x \in \mathcal{R}(A_\lambda^m)^a$, since $x = \psi(1)x = \lim_{\lambda \to 0} A_\lambda^m x$. This and an obvious inclusion $\mathcal{R}(A_\lambda^m)^a \subset \mathcal{R}(A_\lambda)^a = R^a$ give the conclusion.

Next, assume that $x \in \mathcal{N}(A_\lambda^m)$. Then from Theorem 1 (d) it follows that $-x = \lim_{\lambda \to 0} A_\lambda^m x - 1^m x$ belongs to N. This, combined with an obvious inclusion $\mathcal{N}(A_\lambda^m) \supset \mathcal{N}(A_\lambda)$, gives the conclusion.

9. Applications of non-negative operators.

Definition. A linear operator A defined on a subspace $\mathcal{D}(A)$ of X into X is said to be *left* (or *right*) *non-negative* if there exists a positive constant λ_0 such that open interval $(-\lambda_0, 0)$ (or $(-\infty, -\lambda_0)$) is contained in the resolvent set of A and the function $\{\lambda(\lambda I + A)^{-1}\}$ is equicontinuous in $(0, \lambda_0)$ (or (λ_0, ∞)) (cf. [2]).

Theorem 2. *Let A be a left non-negative operator in X. Then, the conditions* (a) $x \in \mathcal{R}(A)^a + \mathcal{N}(A)$, (b) $x \in \mathcal{R}(A)^a$, *and* (c) $x \in \mathcal{N}(A)$ *are equivalent to each one of the conditions* (ii)~(iv) *of part* (b), (c), *and* (d) *in Theorem* 1 *with A_λ replaced by $A(\lambda I + A)^{-1}$, respectively.*

Proof. Since $I - \lambda(\lambda I + A)^{-1} = A(\lambda I + A)^{-1}$, $\mathcal{R}(A(\lambda I + A)^{-1}) = \mathcal{R}(A)$, and $\mathcal{N}(A(\lambda I + A)^{-1}) = \mathcal{N}(A)$, this theorem is a special case of Theorem 1.

Theorem 3. *Let A be a right non-negative operator. Then the following conditions are mutually equivalent;*

(i) $x \in \mathcal{D}(A)^a$,

(ii) $\psi(\lambda(\lambda I + A)^{-1})x \to \psi(1)x$ *as* $\lambda \to +\infty$ *for any polynomial* ψ,

(iii) *There exist a polynomial ψ with $\psi(0) \neq \psi(1)$ and a sequence $\{\lambda_n\}$ which diverges to $+\infty$ such that $\psi(\lambda_n(\lambda_n I + A)^{-1})x$ converges as $n \to \infty$.*

(iv) *There exist a polynomial* ψ, *a point* y *in* X *and a sequence* $\{\lambda_n\}$ *which diverges to* $+\infty$ *as* $n\to\infty$ *such that*

$$x=\lim_{n\to\infty}\psi(\lambda_n(\lambda_n I+A)^{-1})y-\psi(0)y.$$

Proof. This result follows from Theorem 1, since $\mathcal{R}(\lambda(\lambda I+A)^{-1})=\mathcal{D}(A)$, $\mathcal{N}(\lambda(\lambda I+A)^{-1})=\{0\}$.

References

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