95. On the Compactness Criterion for Probability Measures on Banach Spaces

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1. Introduction. A compactness criterion for a set of probability measures on a real separable Hilbert space was given by Prokhorov [11, Theorem 1.14], in terms of their characteristic functionals. In this note we shall prove that a natural generalization of Prokhorov's result to Banach spaces is not valid unless X is isomorphic to a Hilbert space. This is also concerned with author's paper [7].

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2. Preliminaries. Let X be a real separable Banach space, X^* its topological dual space and $\mathcal{B}(X)$ the Borel σ-algebra. By a *random element* in X defined on a basic probability space $(\Omega, \mathcal{A}, P)$ we mean a measurable mapping $(\Omega, \mathcal{A}, P) \to (X, \mathcal{B}(X))$. Every random element ξ induces on $(X, \mathcal{B}(X))$ the probability measure $\mu_\xi = P \circ \xi^{-1}$ which is called its *distribution*. A random element ξ is said to be *Gaussian* if for each $f \in X^*$, $\langle \xi(\cdot), f \rangle$ is a (possibly degenerate) real Gaussian random variables on $(\Omega, \mathcal{A}, P)$.

We identify the set $\mathcal{P}(X)$ of all probability measures on $(X, \mathcal{B}(X))$ with the corresponding subset of $C(X)^*$ under the natural injection $\mu \in \mathcal{P}(X) \to \int_X \varphi(x)\mu(dx)$, $\varphi \in C(X)$, where $C(X)$ is the Banach space of all bounded continuous real functions on X. In this note we define the topology on $\mathcal{P}(X)$ as the relative topology induced by the weak* topology on $C(X)^*$. Then $\mathcal{P}(X)$ is a Polish space (see [11]). For each $\mu \in \mathcal{P}(X)$ the *characteristic functional* of μ is defined by

$$\hat{\mu}(f) = \int_X \exp\{i\langle x, f\rangle\}\mu(dx), \qquad f \in X^*.$$

We shall denote by $\mathcal{N}(X^*, X)$ the Banach space of all nuclear operators from X^* into X with the nuclear norm $\nu(\cdot)$ (see [4] and [12]). A nuclear operator $R: X^* \to X$ is called an *S-operator* if it is positive and symmetric, i.e., $\langle Rf, f\rangle \geq 0$ for all $f \in X^*$ and $\langle Rf, g\rangle = \langle Rg, f\rangle$ for all $f, g \in X^*$. Let ξ be a random element in X satisfying $\int \|\xi(\omega)\|^2 P(d\omega) < \infty$. Then the operator $R_\xi : X^* \to X$ defined by the equality

$$R_\xi f = \int_\Omega \langle \xi(\omega), f\rangle \xi(\omega) P(d\omega)$$

(the integral is understood in the sense of Bochner) is an S-operator, and it is called the *covariance operator* of ξ (see [2]).

Let $(\gamma_n)_{n\geq 1}$ be a sequence of independent standard Gaussian random variables. A Banach space X is said to be *of type* 2 if for every sequence $(x_n)_{n\geq 1}$ in X such that $\sum_{n=1}^{\infty}\|x_n\|^2<\infty$, we have that the series $\sum_{n=1}^{\infty}\gamma_n x_n$ converges a.s. (=almost surely) and is said to be *of cotype* 2 if for every sequence $(x_n)_{n\geq 1}$ such that the series $\sum_{n=1}^{\infty}\gamma_n x_n$ converges a.s., we have $\sum_{n=1}^{\infty}\|x_n\|^2<\infty$. We know that if X is of type 2 and of cotype 2 then it is isomorphic to a Hilbert space (see [9]).

3. Main result. Let us recall that a subset A of a metric space is said to be *relatively compact* if every sequence in A contains a convergent subsequence. Now we can state our result precisely:

Theorem. *For a real separable Banach space X, the following assertions* (a) *and* (b) *are equivalent*:

(a) *X is isomorphic to a Hilbert space.*

(b) *A subset K of $\mathcal{P}(X)$ is relatively compact in $\mathcal{P}(X)$ if* (*and only if*) *for each $\varepsilon>0$, there exists a family $\{S_{\mu,\varepsilon};\mu\in K\}$ of S-operators from X^* into X which satisfies the following two conditions*:

(i) *$1-Re\hat{\mu}(f)\leq\langle S_{\mu,\varepsilon}f,f\rangle+\varepsilon$ for all $f\in X^*$ and all $\mu\in K$.*

(ii) *The set $\{S_{\mu,\varepsilon};\mu\in K\}$ is relatively compact in $\mathcal{N}(X^*,X)$.*

Before starting to prove theorem, we should remark the following: According to [1], the set $\{S_\alpha\}$ of S-operators in $\mathcal{N}(l_q,l_p)(1/p+1/q=1, 2\leq p<\infty)$ is relatively compact if and only if the set $\{\langle S_\alpha e_j,e_j\rangle\}$ is relatively compact in $l_{p/2}$, where $\{e_j\}$ is a natural basis in l_q. For $p=2$, a simple and elementary proof of the above compactness criterion can be found in [6]. Using this criterion, together with Prokhorov's result, we see that the assertion (b) holds when X is a Hilbert space. Therefore we may consider the assertion (b) as a natural generalization of Prokhorov's result to Banach spaces. When $X=l_p$ $(2<p<\infty)$, however, conditions (i) and (ii) of the above theorem are not enough for the validity of the assertion (b) (see [8]).

Proof of Theorem. From the above remark we have only to show the implication (b)⇒(a). According to Kwapień's result [9] stated at the end of preliminaries, it is sufficient to show that X is of type 2 and of cotype 2. Let $(x_j)_{j\geq 1}$ be a sequence in X such that $\sum_{j=1}^{\infty}\|x_j\|^2<\infty$ and we define random elements in X as follows:

(1) $$\xi_n(\cdot)=\sum_{j=1}^{n}\gamma_j(\cdot)x_j,\qquad n\geq 1,$$

where $(\gamma_j)_{j\geq 1}$ is a sequence of independent standard Gaussian random variables. Now let us consider a sequence $(S_n)_{n\geq 1}$ of S-operators in $\mathcal{N}(X^*,X)$ defined by

(2) $$S_n f=\sum_{j=1}^{n}\langle x_j,f\rangle x_j,\qquad f\in X^*,\ n\geq 1.$$

Then the set $\{S_n\}$ is relatively compact in $\mathcal{N}(X^*,X)$. To see this it is enough to show that $(S_n)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{N}(X^*,X)$. From (2) and the definition of the nuclear norm we get for $n>m\geq 1$,

(3) $$\nu(S_n-S_m)\leq\sum_{j=m+1}^{n}\|x_j\|^2.$$

Thus by (3) and the fact that $\sum_{j=1}^{\infty}\|x_j\|^2<\infty$, we see that $(S_n)_{n\geq 1}$ is a Cauchy sequence. Let us denote by μ_n the distribution of ξ_n. Then a routine calculation shows that

$$\hat{\mu}_n(f)=\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\langle x_j, f\rangle^2\right\}=\exp\left\{-\frac{1}{2}\langle S_n f, f\rangle\right\},$$

which implies that

$$(4) \qquad 1-\hat{\mu}_n(f)\leq\langle S_n f, f\rangle \qquad \text{for all } f\in X^*.$$

Thus by (4) and relative compactness of $\{S_n\}$, conditions (i) and (ii) of the assertion (b) are satisfied, and hence we see that the set $\{\mu_n\}$ is relatively compact in $\mathcal{P}(X)$. Consequently, by [5] ξ_n converges a.s., i.e., X is of type 2.

Next we prove that X is of cotype 2. Suppose to the contrary that there exists a sequence $(x_j)_{j\geq 1}$ such that $\sum_{j=1}^{\infty}\gamma_j x_j$ converges a.s., but $\sum_{j=1}^{\infty}\|x_j\|^2=\infty$. (Without loss of generality we may assume that $x_j\neq 0$ for all $j\geq 1$.) If we set $a_k=\sum_{j=1}^{k}\|x_j\|^2$ then $a_k\to\infty$ and also $\sum_{k=1}^{\infty}\|x_k\|^2/a_k=\infty$. According to the idea in [10, Theorem 2.3], we define a sequence $(\zeta_k)_{k\geq 1}$ of independent symmetrically distributed random elements in X with distributions such that

$$(5) \qquad \begin{aligned} &P\left(\zeta_k=-\frac{a_k^{1/2}x_k}{\|x_k\|}\right)=P\left(\zeta_k=\frac{a_k^{1/2}x_k}{\|x_k\|}\right)=-\frac{1}{5}\cdot\frac{\|x_k\|^2}{a_k},\\ &P(\zeta_k=0)=1-\frac{2}{5}\cdot\frac{\|x_k\|^2}{a_k}.\end{aligned}$$

Then, by the Borel-Cantelli lemma, ζ_k does not converge to 0 a.s. so that

$$(6) \qquad \eta_n\equiv\sum_{k=1}^{n}\zeta_k \text{ diverges a.s.}$$

Let us consider a sequence $(S_n)_{n\geq 1}$ defined by (2). A routine calculation shows that each S_n is the covariance operator of a Gaussian random element $\xi_n=\sum_{j=1}^{n}\gamma_j x_j$. Then, since ξ_n converges a.s., by [3, Theorem 1] S_n converges in $\mathcal{N}(X^*, X)$, and hence the set $\{S_n\}$ is relatively compact in $\mathcal{N}(X^*, X)$. Now let us denote by λ_n the distribution of η_n. Then a routine calculation using (5) shows that $\hat{\lambda}_n(f)=\prod_{k=1}^{n}(1-\theta_k)$, where

$$\theta_k=\frac{2}{5}\cdot\frac{\|x_k\|^2}{a_k}\left[1-\cos\left\langle\frac{a_k^{1/2}x_k}{\|x_k\|}, f\right\rangle\right].$$

Since $0\leq\theta_k<1$, we have $\hat{\lambda}_n(f)=\prod_{k=1}^{n}(1-\theta_k)\geq 1-\sum_{k=1}^{n}\theta_k$, which implies that

$$(7) \qquad \begin{aligned} 1-\hat{\lambda}_n(f)&\leq\frac{2}{5}\sum_{k=1}^{n}\frac{\|x_k\|^2}{a_k}\left[1-\cos\left\langle\frac{a_k^{1/2}x_k}{\|x_k\|}, f\right\rangle\right]\\ &\leq\frac{2}{5}\sum_{k=1}^{n}\langle x_k, f\rangle^2\leq\langle S_n f, f\rangle \qquad \text{for all } f\in X^*.\end{aligned}$$

Thus by (7) and the relative compactness of $\{S_n\}$, conditions (i) and (ii) of the assertion (b) are satisfied, and hence we see that the set $\{\lambda_n\}$ is relatively compact in $\mathcal{P}(X)$. Consequently, by [5] η_n converges a.s. and this contradicts (6). The proof is now complete.

References

[1] S. Chevet: Compacité dans l'espace des probabilités de Radon gaussiennes sur un Banach. C. R. Acad. Sci. Paris, ser. 1, **296**, 275–278 (1983).

[2] S. Chevet, S. A. Chobanjan, W. Linde and V. I. Tarieladze: Caractérisation de certaines classes d'espaces de Banach par les mesures gaussiennes. ibid., ser. A, **285**, 793–796 (1977).

[3] S. Chevet: Gaussian measures and large deviations. Lect. Notes in Math., no. 990, 30–46 (1983).

[4] A. Grothendieck: Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc., **16**, 1–191 (1955).

[5] K. Itô and M. Nisio: On the convergence of sums of independent Banach space valued random variables. Osaka J. Math., **5**, 35–48 (1968).

[6] J. Kawabe: Characterization of Hilbert spaces by measure theoretic probability theory. Doctoral Dissertation, Tokyo Institute of Technology (1986).

[7] ——: Probabilistic characterization of certain Banach spaces (to appear in Kodai Math. J.).

[8] J. Kuelbs and V. Mandrekar: Harmonic analysis on certain vector spaces. Trans. Amer. Math. Soc., **149**, 213–231 (1970).

[9] S. Kwapień: Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. Studia Math., **44**, 583–595 (1972).

[10] V. Mandrekar: Characterization of Banach space through validity of Bochner theorem. Lect. Notes in Math., no. 644, 314–326 (1978).

[11] Yu. V. Prokhorov: Convergence of random processes and limit theorems in probability theory. Theor. Probability Appl., **1**, 157–214 (1956).

[12] K. Yosida: Functional Analysis. 6th ed., Springer-Verlag (1981).