94. A Stochastic Differential Equation Arising from the Vortex Problem

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1. Introduction. The purpose of this paper is to solve a stochastic differential equation (SDE) which represents the vortex flow in the *whole plane*.

A system of n vortices $Z_t=(Z_t^1, \cdots, Z_t^n)$ ($Z_t^i \in R^2$ is the position of the i^{th} vortex at time t and $\gamma_i \in R$ its vorticity intensity) in a viscous and incompressible fluid satisfies the following SDE.

(1) $$dZ_t^i=\sigma dB_t^i+\sum_{\substack{j=1\\ j\neq i}}^{n} \gamma_j K(Z_t^i-Z_t^j)dt, \qquad 1\leq i\leq n,$$

where

(2) $$K(z)=\nabla^{\perp}G(z) \qquad z=(x, y) \in R^2,$$

$G(z)=-(2\pi)^{-1}\log|z|$, $\nabla^{\perp}=(\partial/\partial y, -(\partial/\partial x))$, $(B_t^1, \cdots, B_t^n)$ is a $2n$-dim. Brownian motion and σ is a constant which is related to the viscosity. Since the coefficients are singular on the set

$$S=\bigcup_{\substack{i\neq j\\ i,j=1}}^{n} \{(z_i) \in R^{2n}\,;\, z_i=z_j\},$$

it is not easy to solve (1). Let L be the generator of (1):

(3) $$L=\nu\Delta+\sum_{\substack{i\neq j\\ i,j=1}}^{n} \gamma_j(\nabla_i^{\perp}G(z_i-z_j))\cdot\nabla_i$$

where

$$\nu=\frac{1}{2}\sigma^2, \quad \nabla_i=\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \nabla_i^{\perp}=\left(\frac{\partial}{\partial y_i}, -\frac{\partial}{\partial x_i}\right).$$

We can rewrite this as

(4) $$L=\nu\Delta+\sum_{\substack{i\neq j\\ i,j=1}}^{n} \gamma_j\nabla_i^{\perp}\cdot(G(z_i-z_j)\nabla_i).$$

One might expect to apply PDE results by taking advantage of this divergence structure. However, they do not apply to the case considered here, because $G(z_i-z_j)$ has a *log-type* singularity.

The key point of the proof is to observe that L is a differential operator of a *generalized divergence form* defined in Section 2 and apply a result obtained in [3].

The coefficients $K(z_i-z_j)$ are locally Lipschitz continuous on $R^{2n}-S$. Hence (1) is uniquely solvable till Z_t hits S. The problem is to show that Z_t is conservative on $R^{2n}-S$. Now, we state our main theorem.

Theorem. *Let* $\tau=\inf\{t>0 : Z_t \in S\}$. *Then for any* $x \in R^{2n}-S$,

(5) $$P_x\{\tau<\infty\}=0.$$

Remark 1. Such a set up for the motion of n vortices in a viscous and incompressible fluid is due to D. Durr and M. Pulvirenti [1]. From their point of view, the following three choices of the domain D are of interest for physics.

(i) $D=R^2$,

(ii) $D=\bar{T}^2=[-R, R]^2$, and the corresponding G is the Green's function of the Poisson equation with the periodic boundary condition.

(iii) D is a bounded domain with smooth boundary, and G is the Green's function for the Dirichlet boundary condition.

They solved this problem in the case (ii). Their argument needs a finite invariant measure of Z_t, hence it is not available in the case (i).

Remark 2. If the all vorticity intensities γ_i are of the same sign, then S. Takanobu [4] shows the result of Theorem by a probabilistic argument.

2. Diffusion processes associated with generalized divergence form. Let $a_{ij}(x)$, $b_{ij}(x)$ be measurable functions on R^n. Consider a differential operator

$$A=\sum_{i,j=1}^n \nabla_i a_{ij}\nabla_j+\sum_{i,j=1}^n (\nabla_i b_{ij})\nabla_j \qquad \left(\nabla_i=\frac{\partial}{\partial x_i}\right).$$

A is said to be *generalized divergence form* if, for some positive constants λ, μ,

(i) $\lambda^{-1}|\xi|^2\leqq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j$ for any $\xi=(\xi_i)\in R^n$,

(ii) $|a_{ij}|, |b_{ij}|\leqq\mu$, $i, j=1, 2, \cdots, n$,

(iii) $\int_{R^n}\sum_{i,j=1}^n b_{ij}\nabla_i\nabla_j\varphi dx=0$ for all $\varphi\in C_0^2(R^n)$.

We write $A\in G(\lambda, \mu)$ if A satisfies the above conditions, and $A\in G_0(\lambda, \mu)$ if $A\in G(\lambda, \mu)$ and a_{ij}, b_{ij} are smooth. It should be noted that A^* (the adjoint of A with respect to Lebesgue measure) is also of class $G(\lambda, \mu)$ by (iii).

Definition. A continuous function $p(t, x, y)$ on $(0, \infty)\times R^n\times R^n$ is said to be a fundamental solution of $\partial/\partial t-A$ $(A\in G(\lambda, \mu))$ if it satisfies the following conditions:

(i) $p(t, x, y)\geqq 0$ and $\int_{R^n} p(t, x, y)dy=1$ for all $(t, x, y)\in(0, \infty)\times R^n\times R^n$.

(ii) Let $\varphi(x)$ be a continuous function on R^n with compact support, and set $u(t, x)=\int_{R^n} p(t, x, y)\varphi(y)dy$. Then $u(t, x)\to\varphi(x)$ uniformly on R^n as $t\to 0$ and

$$\sum_{i=1}^n\int_0^\infty\int_{R^n}|\nabla_i u(t, x)|^2\,dxdt<\infty,$$

$$\sup_{0\leqq t<\infty}\int_{R^n}|u(t, x)|^2\,dx<\infty,$$

$$\int_0^\infty\int_{R^n}\left\{u\frac{\partial}{\partial t}\psi-\sum_{i,j=1}^n(a_{ij}\nabla_j u\nabla_i\psi-b_{ij}\nabla_i u\nabla_j\psi-b_{ij}u\nabla_i\nabla_j\psi)\right\}dxdt=0$$

for all $\psi(t,x)\in C_0^2((0,\infty)\times R^n)$.

Let $A=\sum_{i,j=1}^n\{\nabla_i a_{ij}\nabla_j+(\nabla_i b_{ij})\nabla_j\}\in G(\lambda,\mu)$. We call a fundamental solution p *regular* if there exists $\{A_k\}_{k=1}^\infty\in G_0(2\lambda,2\mu)$ such that $\lim_{k\to\infty}p^k(t,x,y)=p(t,x,y)$ compact uniformly on $(t,x,y)\in(0,\infty)\times R^n\times R^n$ where p^k is a fundamental solution of $\partial/\partial t-A_k$.

Lemma 1. *Let $A\in G(\lambda,\mu)$. Then there exists a regular fundamental solution. Moreover an arbitrary regular fundamental solution $p(t,x,y)$ satisfies the following:*

$$(6)\qquad (C_1t)^{-\frac{1}{2}n}\exp(-C_2|x-y|^2/t)\leqq p(t,x,y)\leqq(C_3t)^{-\frac{1}{2}n}\exp(-C_4|x-y|^2/t)$$

for all $(t,x,y)\in(0,\infty)\times R^n\times R^n$ with positive constants $C_1,\cdots,C_4$ depending only on λ, μ and n,

$$(7)\qquad |p(t,x,y)-p(t',x',y')|\leqq C_5(|t-t'|^{\frac{1}{2}\alpha}+|x-x'|^\alpha+|y-y'|^\alpha)$$

for all (t,x,y) and $(t',x',y')\in(0,T)\times R^n\times R^n$ with positive constants C_5 and α depending only on T, λ, μ and n,

$$(8)\qquad \int_{R^n}p(s,x,y)p(t,y,z)dy=p(s+t,x,z),$$

$$(9)\qquad \int_{R^n}p(s,x,y)dx=1,$$

$$(10)\qquad \sup_{0\leqq t<\infty}\|u(t,\cdot)\|_{L^2(R^n)}\leqq\|\varphi\|_{L^2(R^2)},\qquad \int_0^\infty\int_{R^n}\sum_{i=1}^n|\nabla_i u|^2\,dxdt\leqq\|\varphi\|_{L^2(R^n)}^2,$$

where

$$u(t,x)=\int_{R^n}p(t,x,y)\varphi(y)dy.$$

See [3] for the proof. A diffusion process $\{X_t\}$ is said to be associated with $A\in G(\lambda,\mu)$ if its transition mechanism is given by a *regular* fundamental solution of $\partial/\partial t-A$. It might happen that plural diffusion processes are associated with A. However, if $A\in G_0(\lambda,\mu)$, we can exclude such a possibility.

3. Proof of Theorem. We first show that L is a generalized divergence form. Put

$$f(z)=-(2\pi)^{-1}xy\log|z|$$
$$a^1(z)=-x^2y^2/\pi|z|^4,\qquad z=(x,y)\in R^2.$$

Then it is easily checked that

$$(11)\qquad G=\nabla_x\nabla_y f+a^1+1/2\pi.$$

Hence

$$(12)\qquad \nabla_xG=\nabla_ya^2+\nabla_xa^1,\qquad \nabla_yG=\nabla_xa^3+\nabla_ya^1,$$

where $a^2=\nabla_x^2f$ and $a^3=\nabla_y^2f$. By (12) and

$$(13)\qquad \begin{cases}|a^1(z)|\leqq 1/4\pi\\ |a^2(z)|=|-3xy/2\pi|z|^2+x^3y/\pi|z|^4|\leqq 3/4\pi\\ |a^3(z)|=|-3xy/2\pi|z|^2+y^3x/\pi|z|^4|\leqq 3/4\pi,\end{cases}$$

we obtain

Lemma 2. $L\in G(\nu,\rho)$ *with* $\rho=(3/4\pi)\sup_j|\gamma_j|$.

By Lemmas 1 and 2, we can conclude that there exists a diffusion process associated with L in the sense of Section 2. Next we shall show only one diffusion process among them satisfies SDE (1). For this purpose we need the following lemma due to Kanda [2].

Lemma 3. *Let $\{X_t\}$ be a time homogeneous diffusion process whose transition probability density $p(t, x, y)$ with respect to Lebesgue measure satisfies*

(14) $(C_6 t)^{-\frac{1}{2}n} \exp(-C_7|x-y|^2/t) \leqq p(t, x, y) \leqq (C_8 t)^{-\frac{1}{2}n} \exp(-C_9|x-y|^2/t)$

$(t, x, y) \in (0, \infty) \times R^n \times R^n$ and $C_6, \cdots, C_9$ are positive constants. Assume $\{X_t\}$ has a dual process $\{X_t\}$ whose transition probability also satisfies (14). *For a Borel set S in R^n, put*

$$\tau_1 = \inf_{t>0} \{X_t \in S\}, \qquad \tau_2 = \inf_{t>0} \{B_t \in S\}$$

(B_t is a Brownian motion in R^n). Then

$$P_x^X(\tau_1 < \infty) = 0 \quad \textit{if and only if} \quad P_x^B(\tau_2 < \infty) = 0.$$

The final step of the proof. Let us first consider an approximation for (1). Set

$$L_k = \nu\Delta + \sum_{\substack{i \neq j \\ i,j=1}}^{n} \gamma_j \{(\nabla_{x_i} a_{ij}^{3,k} + \nabla_{y_i} a_{ik}^{1,k}) \nabla_{x_i} - (\nabla_{x_i} a_{ij}^{1,k} + \nabla_{y_i} a_{ij}^{2,k}) \nabla_{y_i}\}.$$

We assume $L_k \in G_0(\nu, 2\rho)$ and $a_{ij}^{h,k}(z_1, \cdots, z_n) = a^h(z_i - z_j)$ $(h=1, 2, 3)$ if $|z_i - z_j| \geqq 1/k$. Let $p^k(t, x, y)$ be a fundamental solution of $\partial/\partial t - L_k$. Since the coefficients of L_k are smooth, p^k is unique. By Lemma 1, we can choose a subsequence $\{p^{k'}\}$ which converges to a regular fundamental solution $p(t, x, y)$ of $\partial/\partial t - L$. Let Z_t^0 denote the diffusion process determined by p. Then Z_t^0 satisfies the assumption of Lemma 3 and

$$P_x(\sigma < \infty) = 0 \qquad \text{for } x \in R^n - S, \text{ where } \sigma = \inf\{t : Z_t^0 \in S\}.$$

Now, it is clear that

$$Z_{t\wedge\tau} \circ P_x = Z_{t\wedge\sigma}^0 \circ P_x \qquad \text{for } x \in R^n - S,$$

which conclude Theorem.

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