93. *Stability Theorem for Singularly Perturbed Solutions to Systems of Reaction-Diffusion Equations*

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§1. **Introduction.** This note presents a stability theorem to singularly perturbed stationary solutions (SPS) of the systems of nonlinear diffusion equations with a small parameter $\varepsilon>0$:

$$\text{(P)}\quad \begin{array}{l} u_t=\varepsilon^2 u_{xx}+f(u,v) \quad \text{and} \quad v_t=Dv_{xx}+g(u,v), \quad x\in I=(0,1),\ t>0, \\ u_x=0=v_x, \qquad x\in\partial I=\{0,1\}. \end{array}$$

The existence problem of SPS has a rather long history, see, for instance, [3]. For the stability properties of SPS, however, very few works have been known (see [2]). An exception is the work for degenerate case $\varepsilon=0$ of a simple density-dependent diffusion system ([1]). Recent works of the authors ([7] and [8]) show the stability of SPS for large D, where the basic method is a perturbation from the limit of $D\uparrow+\infty$. However, the stability of SPS for a general D has remained open up to the present time. In this note, we give a new idea to solve the stability problem of SPS of one mode type (SPS1) for a general D, where *the singular limit eigenvalue problem* plays a key role. Let us state the main assumptions for f and g. They are smooth functions defined on an open set $\mathcal{O}$ in $\boldsymbol{R}^2$ such that

(A.1) The nullcline of f is sigmoidal, and consists of three curves $u=h_-(v)$, $h_0(v)$, and $h_+(v)$ with $h_-(v)<h_0(v)<h_+(v)$.

(A.2) $J(v)$ has an isolated zero at $v=v^*$ such that $dJ/dv<0$ at $v=v^*$, where

$$J(v)=\int_{h_-(v)}^{h_+(v)} f(s,v)ds.$$

(A.3) Let
$$G(v)=\begin{cases} g(h_-(v),v), & v\leqq v^* \\ g(h_+(v),v), & v\geqq v^*. \end{cases}$$

Then $dG/dv<0$. Moreover, $g>0$ on the curve $\mathcal{C}_+: u=h_+(v)$ for $v\geqq v^*$, and $g<0$ on $\mathcal{C}_-: u=h_-(v)$ for $v\leqq v^*$. Also, $f_u<0$ on $\mathcal{C}_+\cup\mathcal{C}_-$.

(A.4) (Stability Assumption) On $\mathcal{C}_+\cup\mathcal{C}_-$, $g_v<0$.

For the definitions of function spaces $H^k(I)$, $H^k_N(I)$, and $C^2_\varepsilon(I)$, see [5] and [6]. Under (A.1)–(A.3), the following result is known.

Existence Theorem of SPS1 (Mimura-Tabata-Hosono [5] and Ito [4]). *Suppose there exists a monotone increasing solution $V=V^*(x)$ of $DV_{xx}+G(V)=0$ in I with $V_x=0$ on ∂I, for a given $D>0$. Then, there exists a constant $\varepsilon_0>0$ such that* (P) *has an ε-family of* SPS1 $U^\varepsilon=(u(x;\varepsilon), v(x;\varepsilon))$ *for $0<\varepsilon<\varepsilon_0$. U^ε is uniformly bounded in $C^2_\varepsilon\times C^2$, and satisfies*

$$\lim_{\varepsilon\downarrow 0} u(x;\varepsilon)=U^*(x)\overset{\text{def}}{=}\begin{cases} h_-(V^*(x)), & x\in[0,x^*) \\ h_+(V^*(x)), & x\in(x^*,1], \end{cases}$$

uniformly on $I\backslash I_\kappa$ *for any* $\kappa>0$, *and*

$$\lim_{\varepsilon\downarrow 0} v(x;\varepsilon)=V^*(x) \qquad \textit{uniformly on } I.$$

Here x^* *indicates the layer position, uniquely determined by* $V^*(x)=v^*$, *and* $I_\kappa=(x^*-\kappa,\ x^*+\kappa)$.

Since (P) is a system of semi-linear parabolic equations, the stability of U^ε is determined by the spectra of the linearized eigenvalue problem:

$$\text{(LP)} \qquad \begin{aligned} &\varepsilon^2 w_{xx}+f_u^\varepsilon w+f_v^\varepsilon z=\lambda w \quad \text{and} \quad Dz_{xx}+g_u^\varepsilon w+g_v^\varepsilon z=\lambda z, \quad \text{in } I,\\ &w_x=0=z_x \qquad \text{on } \partial I,\end{aligned}$$

where all the partial derivatives are evaluated at U^ε. If $\operatorname{Re}\lambda<0$ for all eigenvalues of (LP), then U^ε is an asymptotically stable solution of (P). It will be convenient to divide the spectrum into two classes; one is *critical eigenvalues* which converge to zero as $\varepsilon\downarrow 0$, and the other *noncritical* ones which are bounded away from zero for small $\varepsilon>0$. Note that noncritical eigenvalues are not dangerous to the stability of U^ε as shown in Lemma 3. Therefore, the stability depends wholly on the asymptotic behavior of critical eigenvalues as $\varepsilon\downarrow 0$. Our conclusion is the following.

Main Theorem. *Under* (A.4), *and* (A.1)–(A.3) *as well, there exists only one critical eigenvalue* $\lambda=\lambda_0(\varepsilon)$, *and which is real and simple. When* $\varepsilon\downarrow 0$, *it behaves as* $\lambda_0(\varepsilon)\simeq -\gamma\varepsilon(\gamma>0)$.

See [9] for the complete proof.

§ 2. Singular limit eigenvalue problem. Let us introduce the *singular limit eigenvalue problem*, which has a Dirac δ-function at the layer position x^*. First, we need the following.

Lemma 1. *Let* $\{\zeta_n^\varepsilon, \phi_n^\varepsilon\}$ *be the complete orthonormal system of Sturm-Liouville problem:*

$$L^\varepsilon\phi \overset{\text{def}}{\equiv} \varepsilon^2\phi_{xx}+f_u^\varepsilon\phi=\zeta\phi \qquad \textit{in } I, \textit{ and } \phi_x=0 \textit{ on } \partial I.$$

Then, the principal eigenvalue ζ_0^ε *is positive for* $\varepsilon>0$, *and tends to zero when* $\varepsilon\downarrow 0$ *as* $\zeta_0^\varepsilon=\varepsilon\hat{\zeta}_0(\varepsilon)$, *where* $\hat{\zeta}_0(\varepsilon)$ *is continuous and* $\hat{\zeta}_0(0)>0$. *All the other eigenvalues remain strictly negative as* $\varepsilon\downarrow 0$, *namely,* $\zeta_n^\varepsilon<-\mu<0$ *for* $n\geqq 1$. *The principal eigenfunction* ϕ_0^ε *is positive, and*

$$\int_I \phi_0^\varepsilon dx=O(\sqrt{\varepsilon}).$$

Since ζ_0^ε can never be an eigenvalue of (LP), it follows that $w=(L^\varepsilon-\lambda)^{-1}(-f_v^\varepsilon z)$ for $\operatorname{Re}\lambda>-\mu$. A substitution of this into the second equation of (LP) leads to

$$(1) \qquad Dz_{xx}+(\zeta_0^\varepsilon-\lambda)^{-1}\langle -f_v^\varepsilon z, \phi_0^\varepsilon\rangle g_u^\varepsilon\phi_0^\varepsilon+g_u^\varepsilon(L^\varepsilon-\lambda)^\dagger(-f_v^\varepsilon z)+g_v^\varepsilon z=\lambda z,$$

where $\langle\ ,\ \rangle$ denotes the inner product in L^2-space and

$$(L^\varepsilon-\lambda)^\dagger(u)=\sum_{n\geqq 1}\frac{\langle u, \phi_n^\varepsilon\rangle}{\zeta_n^\varepsilon-\lambda}\phi_n^\varepsilon.$$

$(L^\varepsilon-\lambda)^\dagger: L^2(I)\to L^2(I)\cap\{\phi_0^\varepsilon\}^\perp$ is uniformly L^2-bounded with respect to ε for $\operatorname{Re}\lambda>-\mu$. One has to derive a limiting equation of (1) as $\varepsilon\downarrow 0$ without losing information about the behavior of λ. The first important step is the following.

Lemma 2. *$(L^\varepsilon-\lambda)^\dagger$ becomes a multiplication operator when $\varepsilon\downarrow 0$. More precisely,*

$$\lim_{\varepsilon\downarrow 0}(L^\varepsilon-\lambda)^\dagger u=u/(f_u^*-\lambda) \qquad \textit{in } L^2\textit{-sense}$$

for any bounded $u\in L^2(I)$ and $\mathrm{Re}\,\lambda>-\mu$, where $f_u^*=f_u(U^*(x), V^*(x))$.

If λ is a noncritical eigenvalue, the second term of the left side of (1) goes to zero in L^2-sense as $\varepsilon\downarrow 0$. Therefore, using (A.3) and (A.4), one obtains an a priori bound for noncritical eigenvalues for small $\varepsilon>0$.

Lemma 3. $\mathrm{Re}\,\lambda<-\delta_0<0$ *for any noncritical eigenvalue λ, where δ_0 is a positive constant which does not depend on ε.*

The next lemma is crucial to derive the singular limit.

Lemma 4.

$$\lim_{\varepsilon\downarrow 0} -f_v^\varepsilon\phi_0^\varepsilon/\sqrt{\varepsilon}=c_1^*\delta^* \qquad \textit{in } H^{-1}\textit{-sense},$$
$$\lim_{\varepsilon\downarrow 0} g_u^\varepsilon\phi_0^\varepsilon/\sqrt{\varepsilon}=c_2^*\delta^* \qquad \textit{in } H^{-1}\textit{-sense},$$

where δ^* is a Dirac δ-function at $x=x^*$, namely, $\delta^*=\delta(x-x^*)$, and c_i^* $(i=1,2)$ are positive constants determined by $-f_v^\varepsilon$ and g_u^ε, respectively.

Let us write the critical eigenvalue λ in the form of $\lambda=\varepsilon\tau(\varepsilon)$, where τ is a continuous function of ε. This scaling will be justified in § 3. Using Lemmas 2 and 4, the limiting equation of (1) called *the singular limit eigenvalue problem* is given by the following weak form:

$$(2) \qquad D\langle z_x, \psi_x\rangle - c_1^*c_2^*(\hat{\zeta}^*-\tau_0)^{-1}\langle z,\delta^*\rangle\langle\delta^*,\psi\rangle-\langle \det^*\cdot f_u^{*-1}z,\psi\rangle=0,$$
$$z\in H_N^1(I), \qquad \text{for any } \psi\in H^1(I),$$

where $\hat{\zeta}^*=\hat{\zeta}_0(0)>0$, $\tau_0=\tau(0)$, and $\det^*=f_u^*g_v^*-f_v^*g_u^*>0$ from (A.3). Hereafter, z will be normalized as $\langle z,\delta^*\rangle=1$. (2) is equivalent to the following form:

$$(3.1) \qquad Dz_{xx}+\det^*\cdot f_u^{*-1}z=0 \qquad \text{in } (0,x^*)\cup(x^*,1) \text{ with } z_x=0 \text{ on } \partial I,$$
$$(3.2) \qquad D[z_x]=-c_1^*c_2^*/(\hat{\zeta}^*-\tau_0),$$

where

$$[z_x]=\lim_{\delta\downarrow 0}\{z_x(x^*+\delta)-z_x(x^*-\delta)\}.$$

Since $\det^*/f_u^*$ is strictly negative from (A.3), the solution z of (3) under $\langle z,\delta^*\rangle=1$ exists uniquely for the appropriate τ_0. We denote this unique solution by z_N^* and τ_0^*. The remaining problem is to determine the sign of τ_0^*. The following observation is a key to judge its sign.

Lemma 5. *Replacing the Neumann boundary conditions $z_x=0$ by Dirichlet conditions $z=0$ in (3), one obtains a new problem denoted by $(3)_D$. Then, there exists a unique solution z_D^* of $(3)_D$ with $\tau_0=0$ under the normalization $\langle z_D^*,\delta^*\rangle=1$.*

Now a comparison of the two solutions (z_N^*,τ_0^*) and $(z_D^*,0)$ leads to the following.

Lemma 6. $[(z_D^*)_x]<[(z_N^*)_x]<0$ *holds, which implies that τ_0^* is strictly negative.*

Consequently, the principal part of the critical eigenvalues is uniquely determined and given by $\lambda_0(\varepsilon)\simeq\tau_0^*\varepsilon$, which leads to the Main Theorem.

§ 3. Justification of the singular limit eigenvalue problem.

Lemma 7. *The inverse operator $K^{\varepsilon,\lambda}$ from $H^{-1}(I)$ to $H_N^1(I)$*

$$K^{\varepsilon,\lambda}=\{-D(d^2/dx^2)-g_u^\varepsilon(L^\varepsilon-\lambda)^\dagger(-f_v^\varepsilon\cdot)-g_v^\varepsilon\cdot+\lambda\cdot\}^{-1},$$

exists for $0\leqq\varepsilon<\varepsilon_0$ *and* $\mathrm{Re}\,\lambda>-\mu$. *Moreover,* $K^{\varepsilon,\lambda}$ *depends continuously on* ε, *and depends analytically on* λ *in operator norm.*

Applying the operator $K^{\varepsilon,\lambda}$ to (1), one sees that (1) has a nontrivial solution z if and only if λ satisfies

$$(4)\qquad \langle K^{\varepsilon,\lambda}(g_u^\varepsilon\phi_0^\varepsilon/\sqrt{\varepsilon}),\ -f_v^\varepsilon\phi_0^\varepsilon/\sqrt{\varepsilon}\rangle=(\zeta_0^\varepsilon-\lambda)/\varepsilon.$$

It follows from Lemmas 4 and 7 that left-hand side of (4) is a continuous function of ε for $0\leqq\varepsilon<\varepsilon_0$ and analytic with respect to λ. Therefore, recalling $\zeta_0^\varepsilon=\varepsilon\hat{\zeta}_0(\varepsilon)$ (Lemma 1), λ must be $O(\varepsilon)$ in order that (5) has a solution $\lambda=\lambda(\varepsilon)$ with $\lambda(0)=0$. Hence, λ can be written as $\lambda=\varepsilon\tau(\varepsilon)$, where τ is a bounded continuous function of ε. Then, one sees that (4) is equivalent to the following scalar equation

$$(5)\qquad \mathcal{F}(\varepsilon,\tau)\overset{\text{def}}{\equiv}\tau-\hat{\zeta}_0+\langle K^{\varepsilon,\varepsilon\tau}(g_u^\varepsilon\phi_0^\varepsilon/\sqrt{\varepsilon}),\ -f_v^\varepsilon\phi_0^\varepsilon/\sqrt{\varepsilon}\rangle=0.$$

Since $\mathcal{F}(0,\tau_0^*)=0$ and $\partial\mathcal{F}/\partial\tau(0,\tau_0^*)=1$, where $\tau_0^*=\hat{\zeta}^*-c_1^*c_2^*\langle K^{0,0}\delta^*,\delta^*\rangle$, one can apply the implicit function theorem to (5), and obtain a unique continuous solution $\tau=\tau(\varepsilon)$ with $\tau(0)=\tau_0^*$. The sign of τ_0^* is strictly negative as in Lemma 6, which concludes the proof of the Main Theorem.

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