# 92. On Some Algebraic Differential Equations with Admissible Algebroid Solutions

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1. **Introduction.** About fifty years ago, K. Yosida ([9]) proved the following theorem.

**Theorem A.** *When the differential equation with rational coefficients*

$$(w')^m=\sum_{j=0}^{p} a_j w^j \Big/ \sum_{k=0}^{q} b_k w^k \qquad (a_p\cdot b_q\neq 0),$$

*where $m$ is a positive integer and $\sum a_j w^j$, $\sum b_k w^k$ are irreducible, admits at least one transcendental $\nu$-valued algebroid solution in $|z|<\infty$, then it holds that*

(1) $$\max(p, q+2m)\leqq 2m\nu.$$

This theorem was extended by several authors ([1], [2], [3], [4] etc.). In this paper, we shall consider the differential equation

(2) $$\Omega(w, w', \cdots, w^{(n)})=P(w)/Q(w),$$

where $\Omega(w, w', \cdots, w^{(n)})=\sum_{\lambda\in I} c_\lambda w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$ $(n\geqq 1)$ is a differential polynomial with meromorphic coefficients, $I$ being a finite set of multi-indices $\lambda=(i_0, i_1, \cdots, i_n)$, ($i_l$ : non-negative integers), for which $c_\lambda\neq 0$, and where $P(w)$, $Q(w)$ are polynomials in $w$ with meromorphic coefficients and mutually prime over the field of meromorphic functions:

$$P(w)=\sum_{j=0}^{p} a_j w^j \quad (a_p\neq 0), \qquad Q(w)=\sum_{k=0}^{q} b_k w^k \quad (b_q\neq 0).$$

The term "meromorphic" (resp. "algebroid") will mean meromorphic (resp. algebroid) in the complex plane. Put

$$\Delta=\max_{\lambda\in I}\sum_{j=0}^{n}(j+1)i_j,\quad \Delta_o=\max_{\lambda\in I}\sum_{j=1}^{n} j i_j,\quad d=\max_{\lambda\in I}\sum_{j=0}^{n} i_j$$

and

$$\sigma=\max_{\lambda\in I}\sum_{j=1}^{n}(2j-1)i_j.$$

An algebroid solution $w=w(z)$ of (2) is said to be admissible when $T(r,f)=S(r,w)$ for all coefficients $f=c_\lambda$, $a_j$ and $b_k$ in (2), where $S(r,w)$ is any quantity satisfying $S(r,w)=o(T(r,w))$ as $r\to\infty$, possibly outside a set of $r$ of finite linear measure.

Recently, Gackstatter and Laine ([1], [2]), Y. He and X. Xiao ([3]) extended Theorem A as follows:

"*If the differential equation* (2) *admits an admissible algebroid solution $w=w(z)$ with $\nu$ branches, then*

(i) $q\leqq 4\Delta_o(\nu-1)$, $p\leqq \Delta+4\Delta_o(\nu-1)$ ([1], [2]),

(ii) $q\leqq 2\sigma(\nu-1)$, $p\leqq q+d+\Delta_o\nu(1-\theta(w,\infty))$ ([3])

*where* $\theta(w,\infty)=1-\limsup_{r\to\infty}\overline{N}(r,w)/T(r,w)$."

In this paper, we shall improve these results and give some examples.

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We use the standard notation of the Nevanlinna theory of meromorphic functions ([5]) or algebroid functions ([6], [7], [8]).

**2. Lemmas.** We shall give some lemmas and a notation here.

**Lemma 1.** *Let $w$ be a nonconstant algebroid function, then*

$$m(r, w^{(n)}/w)=S(r, w) \quad (n=1, 2, \cdots) \qquad \text{(see [8]).}$$

We can easily prove this lemma as in the case of meromorphic functions (see [5], p. 115) using the inequalities (20) and (21) in [8].

**Lemma 2.** *Let $P(w)$ and $Q(w)$ be as in § 1 and $w=w(z)$ be an algebroid function such that $T(r, a_j)=S(r, w)$ $(0\leqq j\leqq p)$ and $T(r, b_k)=S(r, w)$ $(0\leqq k\leqq q)$. Then,*

$$T(r, P/Q)=\max(p, q)T(r, w)+S(r, w) \qquad \text{([1], [4]).}$$

Let $w=w(z)$ be a $\nu$-valued algebroid function and $a$ be a pole of $w$. Then, in a neighbourhood of $a$, we have the following expansions of $w$:

$$w(z)=(z-a)^{-\tau_i/\lambda_i}S((z-a)^{1/\lambda_i}),$$

where $i=1, 2, \cdots, \mu(a)$ $(\leqq \nu)$, $1\leqq \tau_i$, $1\leqq \lambda_i$, $\sum \lambda_i=\nu$ and $S(t)$ is a regular power series of $t$ such that $S(0)\neq 0$. Put

$$n_b(r, w)=\sum_{|a|\leqq r}\sum_{i=1}^{\mu(a)}(\lambda_i-1)$$

and

$$\nu N_b(r, w)=\int_0^r (n_b(t, w)-n_b(0, w))/t\,dt+n_b(0, w)\log r.$$

It is trivial that

$$(3) \qquad N_b(r, w)\leqq(\nu-1)\overline{N}(r, w).$$

**3. Theorem.** We use the same notation as in §§ 1–2.

**Theorem.** *If the differential equation (2) admits an admissible algebroid solution $w=w(z)$ with $\nu$ branches, then*

$$\max(p, q+\Delta)\leqq \Delta+\sigma\xi$$

*and*

$$p\leqq \min\{q+d+\Delta_0(1-\theta(w, \infty)+\xi(w, \infty)),\ \Delta+\sigma\xi\},$$

*where $\xi=\limsup_{r\to\infty} N(r, \mathfrak{X})/T(r, w)$ (the ramification index of the Riemann surface of $w$) and $\xi(w, \infty)=\limsup_{r\to\infty} N_b(r, w)/T(r, w)$.*

*Proof.* Let $\alpha\neq 0$ be a constant such that $P(\alpha)\neq 0$ and $Q(\alpha)\neq 0$. This is possible as $a_p\cdot b_q\neq 0$. Substituting $w=w(z)$ in (2) and dividing by $(w(z)-\alpha)^d$, we have the relation

$$(4) \qquad \Omega(w, w', \cdots, w^{(n)})/(w-\alpha)^d=P(w)/(w-\alpha)^d Q(w).$$

Note that $P(w(z))\not\equiv 0$ and $Q(w(z))\not\equiv 0$ as $w=w(z)$ is admissible, and that $P(w)$, $(w-\alpha)^d Q(w)$ are mutually prime by the choice of $\alpha$. Here, we estimate the $T$-function of both sides of (4). By Lemma 1,

$$(5) \qquad m(r, \Omega/(w-\alpha)^d)\leqq dm(r, 1/(w-\alpha))+S(r, w).$$

We denote by $\tau(c, f)$ the order of pole of $f$ at $z=c$.

(i) When $c$ is not a pole of $w$,

$$(6) \qquad \tau(c, \Omega/(w-\alpha)^d)\leqq \tau(c, 1/(w-\alpha)^d)+\tau(c, 1/Q(w))+\sum \tau(c, a_j).$$

(ii) When $c$ is a pole of $w$,

$$\tau(c, (w^{(l)}/(w-\alpha))^{i_l})=\tau(c, ((w-\alpha)^{(l)}/(w-\alpha))^{i_l})=\mu l i_l \qquad (l\geqq 1),$$

where $w-\alpha=(z-c)^{-\tau/\mu}S((z-c)^{1/\mu})$ near $z=c$ $(\mu\geqq 1,\ \tau\geqq 1)$. Therefore,

$$\tau(c, c_\lambda w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}/(w-\alpha)^d)\leqq\mu\textstyle\sum_{l=1}^n li_l+\tau(c, c_\lambda)$$

so that

(7) $\tau(c, \Omega/(w-\alpha)^d)\leqq\Delta_0\mu+\sum\tau(c, c_\lambda)=\Delta_0+\Delta_0(\mu-1)+\sum\tau(c, c_\lambda).$

From (6) and (7), we obtain

(8) $N(r, \Omega/(w-\alpha)^d)\leqq dN(r, 1/(w-\alpha))+N(r, 1/Q)$
$+\Delta_0(\bar{N}(r, w)+N_b(r, w))+S(r, w).$

As $N(r, 1/Q)\leqq T(r, Q)+O(1)=qT(r, w)+S(r, w)$, using Lemma 2 and combining (5) and (8), we obtain

(9) $T(r, \Omega/(w-\alpha)^d)\leqq(q+d)T(r, w)+\Delta_0\bar{N}(r, w)+\Delta_0 N_b(r, w)+S(r, w).$

On the other hand, by Lemma 2

(10) $T(r, P/(w-\alpha)^dQ)=\max(p, q+d)T(r, w)+S(r, w).$

From (4), (9) and (10), we obtain

$$\max(p, q+d)T(r, w)\leqq(q+d)T(r, w)+\Delta_0\bar{N}(r, w)+\Delta_0 N_b(r, w)+S(r, w),$$

from which we easily have

(11) $p\leqq q+d+\Delta_0(1-\theta(w, \infty)+\xi(w, \infty)).$

Next, put $w-\alpha=1/v$ in (2). Then, as

$$w^{(j)}=(1/v)^{(j)}=H_j(v, v', \cdots, v^{(j)})/v^{j+1} \qquad (j=1, 2, \cdots),$$

where $H_j$ is a homogeneous polynomial of degree $j$ in $v, \cdots, v^{(j)}$, and

$$c_\lambda w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}=c_\lambda(\alpha v+1)^{i_0}H_1^{i_1}\cdots H_n^{i_n}v^{-(i_0+2i_1+\cdots+(n+1)i_n)},$$

the differential equation (2) becomes

(12) $H(v, v', \cdots, v^{(n)})=\tilde{P}(v)/\tilde{Q}(v),$

where $\deg\tilde{P}=p$, $\deg\tilde{Q}=p-\Delta$ when $q\leqq p-\Delta$ and $\deg\tilde{P}=q+\Delta$, $\deg\tilde{Q}=q$ when $q>p-\Delta$,

$$H(v, v', \cdots, v^{(n)})=\textstyle\sum_{\lambda\in I}c_\lambda(\alpha v+1)^{i_0}H_1^{i_1}\cdots H_n^{i_n}v^{\Delta-(i_0+2i_1+\cdots+(n+1)i_n)}.$$

We estimate $T(r, H)$. First, by Lemma 1 we can easily obtain

(13) $m(r, H)\leqq\Delta m(r, v)+S(r, v).$

Next, we estimate $N(r, H)$.

(i) When $c$ is not a pole of $v$,

$$\tau(c, H)\leqq\Delta_0(\mu-\tau)^+ +\textstyle\sum\tau(c, c_\lambda)\leqq\Delta_0(\mu-1)+\sum\tau(c, c_\lambda),$$

where $v(z)=v(c)+(z-c)^{\tau/\mu}S((z-c)^{1/\mu})$ near $z=c$ $(\mu\geqq2, \tau\geqq1)$.

(ii) When $c$ is a pole of $v$ of order $\tau$ and not a branch point, as

$$\tau(c, c_\lambda(\alpha v+1)^{i_0}H_1^{i_1}\cdots H_n^{i_n}v^{\Delta-(i_0+2i_1+\cdots+(n+1)i_n)})\leqq\tau\Delta+\tau(c, c_\lambda),$$
$$\tau(c, H)\leqq\tau\Delta+\textstyle\sum\tau(c, c_\lambda).$$

(iii) When $c$ is a pole of $v$ and a branch point, as

$$\tau(c, c_\lambda(\alpha v+1)^{i_0}H_1^{i_1}\cdots H_n^{i_n}v^{\Delta-(i_0+2i_1+\cdots+(n+1)i_n)})$$
$$\leqq\tau\Delta+(\mu-1)\textstyle\sum_{j=1}^n(2j-1)i_j+\tau(c, c_\lambda)\leqq\tau\Delta+(\mu-1)\sigma+\tau(c, c_\lambda),$$

where $v(z)=(z-c)^{-\tau/\mu}S((z-c)^{1/\mu})$ near $z=c$ $(\mu\geqq2, \tau\geqq1)$,

$$\tau(c, H)\leqq\tau\Delta+(\mu-1)\sigma+\textstyle\sum\tau(c, c_\lambda).$$

From (i), (ii) and (iii), we obtain the inequality

(14) $N(r, H)\leqq\Delta N(r, v)+\sigma N(r, \mathcal{X})+S(r, v).$

Using the inequality obtained from (12), (13), (14) and by Lemma 2, we obtain the inequality

(15) $\max(p, q+\Delta)\leqq\Delta+\sigma\xi \qquad (\leqq\Delta+2\sigma(\nu-1)).$

Combining (11) and (15), we have the second inequality of Theorem.

**Remark.** By (3), we have the inequality

$$\Delta_o(1-\theta(w,\infty)+\xi(w,\infty))\leqq\nu\Delta_o(1-\theta(w,\infty)).$$

**Corollary.** *If* $\xi=0$, *then* $q=0$, $p\leqq\min\{\Delta, d+\Delta_o(1-\theta(w,\infty)\}$.

This is an extension of Theorem 4 in [9].

**4. Examples.** We shall give some examples which show that our theorem is better than that of He and Xiao.

**Example 1.** The algebroid function $w$ defined by $w^{2m}+2-1/\cos^2 z=0$ $(m\geqq 1)$ is an admissible solution of the differential equation

$$(w')^2=(w^{6m}+5w^{4m}+8w^{2m}+4)/m^2w^{4m-2}.$$

In this case, $q+d+\nu\Delta_o(1-\theta(w,\infty))=8m>p$ and $q+d+\Delta_o(1-\theta(w,\infty)+\xi(w,\infty))=\Delta+\sigma\xi=6m=p$.

**Example 2.** The algebroid function $w$ defined by $w^{2m}-3\tan^2 z+z-2=0$ $(m\geqq 1)$ is an admissible solution of the differential equation

$$mw^{2m-1}(w')^2+w'=(4w^{6m}+12zw^{4m}+12(z^2-1)w^{2m}+4z^3-12z-11)/12mw^{2m-1}.$$

In this case,

$$q+d+\nu\Delta_o(1-\theta(w,\infty))=8m>p,\qquad \Delta+\sigma\xi=8m-1$$

and

$$q+d+\Delta_o(1-\theta(w,\infty)+\xi(w,\infty))=6m=p.$$

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