90. The Number of Embeddings of Integral Quadratic Forms. I*)

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1. Introduction. An *integral quadratic form* is a free Z-module L of finite rank together with a map $q: L \to Z$ such that the induced map $b: L \times L \to Z$ defined by $b(x, y) = q(x+y) - q(x) - q(y)$ is Z-bilinear, and such that $q(nx) = n^2 q(x)$ for all $n \in Z$ and $x \in L$. (This is sometimes called an "even" form in the literature, since the function $x \mapsto b(x, x) = 2q(x)$ assumes only even values.) The adjoint map of the associated bilinear form b is a Z-linear map $\mathrm{Ad}\, b: L \to L^{\#} = \mathrm{Hom}\,(L, Z)$; L is called *nondegenerate* if $\mathrm{Ad}\, b$ is injective, and *unimodular* if $\mathrm{Ad}\, b$ is an isomorphism.

If M and L are integral quadratic forms, an *embedding* of M into L is an injective homomorphism of Z-modules $\phi: M \to L$ which preserves the quadratic maps; ϕ is called *primitive* if $\mathrm{coker}\, \phi$ is free. Nikulin [3] has given necessary and sufficient conditions for the existence of a primitive embedding of M into L in the case that M is nondegenerate and L is indefinite and unimodular.

A Z-module isomorphism $\sigma: M \to L$ which preserves the quadratic maps is called an *isometry*. The group of all isometries from L to itself is denoted by $O(L)$. We say that two primitive embeddings $\phi_1, \phi_2: M \to L$ are *equivalent* if there is an isometry σ in $O(L)$ such that $\sigma \circ \phi_1 = \phi_2$. (There are also some restricted notions of equivalence in which σ is required to lie in a specified subgroup of $O(L)$.) Our goal is to count the number of equivalence classes of primitive embeddings from a nondegenerate M into an indefinite unimodular L, assuming that one such embedding exists. In this note, we modify some arguments of Nikulin [3] and Wall [4] to express this number in terms of a certain invariant of the orthogonal complement N of M in L; a subsequent note will give a procedure for computing that invariant when N is indefinite and has rank at least three. The proofs, together with some applications to algebraic geometry, will be given elsewhere.

2. Real quadratic forms and subgroups of the orthogonal group. Let L be a nondegenerate integral quadratic form. If we extend q to a map $q: L \otimes R \to R$ by the requirement $q(rx) = r^2 q(x)$ for $r \in R$ and $x \in L$, then

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$L\otimes R$ becomes a nondegenerate real quadratic form. Such a form has a *signature* (r_+, r_-), where r_+ is the number of positive eigenvalues and r_- is the number of negative eigenvalues; we also call this the signature of L. Of course, $r=r_+ + r_-$ is the rank of L since L is nondegenerate. L is *indefinite* if $r_+r_- \neq 0$.

Let $\{+, -\}$ be a group with two elements, with identity $+$. We define two natural homomorphisms det, spin : $O(L\otimes R) \to \{+, -\}$ as follows : $\det(\sigma)$ is the sign of the determinant of σ, and $\mathrm{spin}(\sigma)$ is the sign of the real spinor norm of σ. More precisely, since the Cartan-Dieudonné theorem guarantees that $O(L\otimes \boldsymbol{R})$ is generated by reflections in non-isotropic elements, if σ is the reflection in $x \in L\otimes \boldsymbol{R}$, then $\det(\sigma) = -$, while $\mathrm{spin}(\sigma) = +$ (or $-$) if $q(x)>0$ (or $q(x)<0$).

These homomorphisms lead to four natural subgroups of $O(L\otimes \boldsymbol{R})$. Define $O_{++}(L\otimes \boldsymbol{R})$ to be the kernel of (det, spin). For $\alpha, \beta \in \{+, -\}$ with $(\alpha, \beta) \neq (+, +)$ we then define a group $O_{\alpha\beta}(L\otimes \boldsymbol{R})$ containing $O_{++}(L\otimes \boldsymbol{R})$ by specifying that $O_{\alpha\beta}(L\otimes \boldsymbol{R})/O_{++}(L\otimes \boldsymbol{R})$ be generated by any element $\sigma \in O(L\otimes \boldsymbol{R})$ such that $\det(\sigma)=\alpha$ and $\mathrm{spin}(\sigma)=\beta$. (If no such element exists, which can only happen if L is definite, we set $O_{\alpha\beta}(L\otimes \boldsymbol{R}) = O_{++}(L\otimes \boldsymbol{R})$.) Note that the index $[O_{\alpha\beta}(L\otimes \boldsymbol{R}) : O_{++}(L\otimes \boldsymbol{R})]$ is either 1 or 2. Finally, for any $\alpha, \beta \in \{+, -\}$, we define $O_{\alpha\beta}(L) = O(L) \cap O_{\alpha\beta}(L\otimes \boldsymbol{R})$.

Let L be a nondegenerate integral quadratic form with signature (r_+, r_-). Following Looijenga and Wahl [1], we define a *positive* (or *negative*) *sign structure* on L to be a choice of one of the connected components of {oriented subspaces $\Pi \subset L\otimes \boldsymbol{R}$ of dimension r_+ (or r_-) such that $q|_\Pi$ is positive (or negative) definite}. (Note that this set has at most two connected components, and that it has exactly two unless L is negative (or positive) definite.) A *total sign structure* on L is a choice of both a positive and a negative sign structure; such a choice determines an orientation on L as well, by the rule that if Π_+ (or Π_-) belongs to the positive (or negative) sign structure, then $\Pi_+ \wedge \Pi_-$ defines the orientation.

Lemma (cf. [1]). *Let L be a nondegenerate integral quadratic form with total sign structure. Then*

(i) $O_{++}(L) = \{\sigma \in O(L) \mid \sigma$ *preserves the total sign structure*$\}$.

(ii) $O_{+-}(L) = \{\sigma \in O(L) \mid \sigma$ *preserves the orientation*$\}$.

(iii) $O_{-+}(L) = \{\sigma \in O(L) \mid \sigma$ *preserves the negative sign structure*$\}$.

(iv) $O_{--}(L) = \{\sigma \in O(L) \mid \sigma$ *preserves the positive sign structure*$\}$.

(v) *If L is indefinite and unimodular, the groups $O(L)$, $O_{++}(L)$, $O_{+-}(L)$, $O_{-+}(L)$, and $O_{--}(L)$ are pairwise distinct.*

Due to this Lemma, we refer to a total sign structure, orientation, negative sign structure and positive sign structure as a $(+, +)$-structure, $(+, -)$-structure, $(-, +)$-structure and $(-, -)$-structure, respectively.

If L_1 and L_2 are two integral quadratic forms on which an (α, β)-structure has been fixed for some $\alpha, \beta \in \{+, -\}$, we call $\sigma : L_1 \to L_2$ an (α, β)-*isometry* if σ is an isometry preserving the (α, β)-structure. We also say that two primitive embeddings $\phi_1, \phi_2 : M \to L$ are *(α, β)-equivalent* if there is some $\sigma \in O_{\alpha\beta}(L)$ such that $\sigma \circ \phi_1 = \phi_2$; these are the restricted versions of equivalence alluded to in section 1.

3. The discriminant-form construction. Let N be a nondegenerate integral quadratic form. The *discriminant-group* of N is the finite abelian group $G_N = \mathrm{Coker}\,(\mathrm{Ad}\, b)$. This group inherits a $\boldsymbol{Q}/\boldsymbol{Z}$-valued quadratic form from the form q on N by the following prescription: for any $\xi \in N^\#$ there is some $n \in \boldsymbol{Z}$ and $x \in N$ such that $n\xi = \mathrm{Ad}\, b(x)$; one defines $q(\xi \bmod N) = n^{-2}q(x)$, and checks that this is well-defined mod $\boldsymbol{Z}$ (cf. [3]). G_N, equipped with this quadratic form, is called the *discriminant-form* of N.

An *isometry* between the discriminant-forms of N_1 and N_2 is an isomorphism of finite groups $\psi : G_{N_1} \to G_{N_2}$ preserving the quadratic forms; we say that N_1 and N_2 have isometric discriminant-forms if such an isometry exists. The group of self-isometries of G_N is denoted by $O(G_N)$. There is a natural homomorphism $O(N) \to O(G_N)$ induced by the adjoint map.

Let $g(N)$ denote the set of isometry classes of nondegenerate integral quadratic forms which have the same signature as N and whose discriminant-forms are isometric to G_N; this is a refinement of the classical notion of the "genus" of an integral quadratic form. If an (α, β)-structure has been fixed on N (for some $\alpha, \beta \in \{+, -\}$), we may use a weaker equivalence relation, and define $g_{\alpha\beta}(N)$ to be the set of (α, β)-isometry classes of nondegenerate integral quadratic forms with a fixed (α, β)-structure, which have the same signature as N and whose discriminant-form is isometric to G_N.

Theorem. *Let $\phi : M \to L$ be a primitive embedding of nondegenerate integral quadratic forms such that L is indefinite and unimodular, and let N be the orthogonal complement of the image of ϕ. Then there are exactly*

$$e(N) = \sum_{N' \in g(N)} [O(G_{N'}) : \mathrm{Image}\,(O(N') \to O(G_{N'}))]$$

equivalence classes of primitive embeddings of M into L, and exactly

$$e_{\alpha\beta}(N) = \sum_{N' \in g_{\alpha\beta}(N)} [O(G_{N'}) : \mathrm{Image}\,(O_{\alpha\beta}(N') \to O(G_{N'}))]$$

(α, β)-equivalence classes of such embeddings. (The sums run over a set of representatives for the equivalence classes.)

Proof. Fix total sign structures on M and on L. The primitive embedding $\phi : M \to L$ determines a total sign structure on the orthogonal complement N by the requirement that if Π_M belongs to the positive (or negative) sign structure on M, and Π_N belongs to the positive (or negative) sign structure on N, then $\phi(\Pi_M) \wedge \Pi_N$ belongs to the positive (or negative) sign structure on L.

If $\phi' : M \to L$ is another primitive embedding with orthogonal complement N', a similar construction produces a total sign structure on N'. Moreover, N' has the same signature as N, since both are equal to the difference of the signatures of L and of M.

The analysis of Nikulin (Proposition 1.6.1 in [3]; cf. also Wall [4]) now shows that ϕ and ϕ' determine isomorphisms of finite groups $\psi: G_M \to G_N$ and $\psi': G_M \to G_{N'}$ which reverse the signs on the quadratic maps, i.e., $q_N \circ \psi = -q_M = q_{N'} \circ \psi'$. Hence, $\psi' \circ \psi^{-1}: G_N \to G_{N'}$ is an isometry, so that N' determines classes in $g(N)$ and $g_{\alpha\beta}(N)$.

Conversely, if we are given $N' \in g(N)$ or $N' \in g_{\alpha\beta}(N)$, there exist isomorphisms $\psi': G_M \to G_{N'}$ which reverse the sign on the quadratic map. A choice of such an isomorphism ψ' then determines a unimodular integral quadratic form $L' \supset M \oplus N'$, which inherits a total sign structure from the construction above. Since L' is indefinite, unimodular, and has the same signature as L, a theorem of Milnor [2] guarantees that there is an isometry $\tau: L \to L'$; by part (v) of the Lemma, we may choose τ to preserve the (α, β)-structure for any $\alpha, \beta \in \{+, -\}$. The composition of τ with the inclusion of M into L' gives a primitive embedding $\phi': M \to L$, which preserves the (α, β)-structure if $N' \in g_{\alpha\beta}(N)$.

It remains to decide when the embeddings $\phi_i: M \to L$ determined by two pairs (N_i, ψ_i) $(i=1, 2)$ are equivalent or (α, β)-equivalent. If $\sigma: L \to L$ represents an equivalence (or (α, β)-equivalence), then σ induces an isometry of N_1 with N_2 (which preserves the (α, β)-structure in the case of (α, β)-equivalence); hence, a necessary condition is that N_1 and N_2 belongs to the same class in $g(N)$ (or $g_{\alpha\beta}(N)$).

If we choose an isometry (or (α, β)-isometry) $\rho: N_1 \to N_2$, then Nikulin's analysis also shows that $\mathrm{id}_M \oplus \rho$ extends to an isometry of L_1 with L_2 if and only if $\psi_1 \circ \rho^{\#} \circ \psi_2 = \mathrm{id}_{G_M}$, where $\rho^{\#}: N_2^{\#} \to N_1^{\#}$ is the induced map; clearly the extension of $\mathrm{id}_M \oplus \rho$ is an (α, β)-isometry if and only if ρ is. Hence the set of equivalence classes (or (α, β)-equivalence classes) of primitive embeddings with orthogonal complement isometric (or (α, β)-isometric) to N' coincides with the set of cosets $O(G_{N'})/\mathrm{Image}(O(N') \to O(G_{N'}))$ (or $O(G_{N'})/\mathrm{Image}(O_{\alpha\beta}(N') \to O(G_{N'}))$), and the theorem follows.

References

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