87. A Note on the Spaces of Self Homotopy Equivalences for Fibre Spaces

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1. Introduction. Throughout this note, we shall work within the category of compactly generated Hausdorff spaces which will be simply called *spaces*. Let X and Y be spaces with base points x_0 and y_0 respectively. We denote by $\mathrm{map}(X, Y)$ and $\mathrm{map}_0(X, Y)$ the space of maps of X to Y and the space of maps of (X, x_0) to (Y, y_0) respectively. Moreover, when k is a map of X to Y, we denote by $\mathrm{map}(X, Y; k)$ the path component of k in $\mathrm{map}(X, Y)$, and $\mathrm{map}_0(X, Y; k)$ is defined similarly. A CW complex means a connected CW complex with non-degenerate base point. Let X be a CW complex with base point x_0, $G(X)$ the space of self homotopy equivalences of X and $G_0(X)$ the space of self homotopy equivalences of (X, x_0). In previous papers [5], [6], [7] we studied $G_0(X)$ when $X=E$ is a fibre space of a fibration: $F\xrightarrow{i}E\xrightarrow{p}B$. This paper is also concerned with $G_0(X)$ for a fibre space X.

2. Main results. We quote the following two theorems [5, 6].

Theorem A. *Let E and B be CW complexes and $p: E\to B$ a fibration with fibre F. Let $n>1$ be a given integer. If F is $(n-1)$-connected and $\pi_i(B)=0$ for every $i\geqq n$, then we have the following fibration:*

$$\mathcal{G}(E \bmod F)\longrightarrow G_0(E)\xrightarrow{\rho} G_0(B)\times G_0(F),$$

where $\mathcal{G}(E \bmod F)$ is the space of self fibre homotopy equivalences of E leaving the fibre F fixed.

Theorem B. *Under the same hypothesis as above, the image of ρ: $G_0(E)\to G_0(B)\times G_0(F)$ is just the union of the path components in $G_0(B)\times G_0(F)$ each of which contains (g, h) satisfying*

$$[\chi_\infty(h)]\circ[k]=[k]\circ[g],$$

where $\chi_\infty(h)$ is a self map of (B_∞, b_∞) and $k: (B, b_0)\to(B_\infty, b_\infty)$ is a classifying map in Allaud's sense for the fibration: $F\xrightarrow{i}E\xrightarrow{p}B$.

Let $\varepsilon(X)$ denote the group $\pi_0(G_0(X))$ for a CW complex X and let R be a subgroup of $\varepsilon(B)\times\varepsilon(F)$ consisting of the elements $([g], [h])$ satisfying $[\chi_\infty(h)]\circ[k]=[k]\circ[g]$. Then our main result is the following

Theorem 1. *Let E and B be CW complexes and $F\xrightarrow{i}E\xrightarrow{p}B=K(\pi, n)$ a fibration classified by a map $k: (B, b_0)\to(B_\infty, b_\infty)$ in Allaud's sense. Let $n>1$ be a given integer. If F is n-connected and $\pi_j(F)=0$ for every $j\geqq 2n$, then we have the following fibration:*

$$\mathrm{map}_0(B, G(F)) \longrightarrow G_0(E) \xrightarrow{\rho} R \times G_{0i}(F),$$

where G_{0i} denotes the path component in $G_0(F)$ containing the identity map id_F, and we have the following exact homotopy sequence of the above fibration for every $j \geqq 0$

$$1 \longrightarrow \pi_j(\mathrm{map}_0(B, G(F))) \longrightarrow \pi_j(G_0(E)) \longrightarrow \pi_j(R \times G_{0i}(F)) \longrightarrow 1.$$

By using the fact that $G(F)$ has the same weak homotopy type as $F \times G_0(F)$ we can easily see the following corollary, which is a generalization of Nomura's theorem (Theorem 3.2 in [3]).

Corollary. *Under the same hypothesis as Theorem 1, we have the following exact sequence*

$$1 \longrightarrow [B, F]_0 \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1.$$

3. Sketch of proof. We shall denote by $X \underset{w}{\simeq} Y$ that X has the same weak homotopy type as Y. First we show the following

Lemma 2. *It holds that*

$$\mathcal{G}(E \bmod F) \underset{w}{\simeq} \mathrm{map}_0(B, F) \underset{w}{\simeq} \mathrm{map}_0(B, G(F)).$$

In fact, since we may regard F as a loop space from our hypothesis (see Corollary 9.9 in [4]), there exists an H-map $\sigma : F \to G(F)$ such that σ induces isomorphisms $\sigma_* : \pi_i(F) \to \pi_i(G(F))$ for every $i \geqq n-1$ by using Theorem 5.1 in [1]. Let B'_∞ and B''_∞ be an $(n-1)$-connective CW complex (B_∞, n) of B and an $(n-1)$-stage Postnikov complex of B_∞ respectively. Then we have a fibration $B'_\infty \xrightarrow{Bj} B_\infty \xrightarrow{\pi} B''_\infty$. By using Theorem 7 in [5] we have the following

$$\mathcal{G}(E \bmod F) \underset{w}{\simeq} \Omega\, \mathrm{map}_0(B, B_\infty; k) \underset{w}{\simeq} \Omega\, \mathrm{map}_0(B, B'_\infty; k'),$$

where $Bj \circ k' \simeq k$. Furthermore, noting that B'_∞ has the same weak homotopy type as BF and B'_∞ itself has the homotopy type of a loop space, we have

$$\begin{aligned}\Omega\, \mathrm{map}_0(B, B'_\infty; k') &\underset{w}{\simeq} \mathrm{map}_0(B, \Omega B'_\infty) \underset{w}{\simeq} \mathrm{map}_0(B, \Omega BF) \\ &\underset{w}{\simeq} \mathrm{map}_0(B, F) \underset{w}{\simeq} \mathrm{map}_0(B, G(F)).\end{aligned}$$

Next we see easily the following

Proposition 3. *Let X be a CW complex and Y a path connected H-space. Then there exists a cross-section $s : Y \to \mathrm{map}(X, Y; l)$ for the following fibration:*

$$\mathrm{map}_0(X, Y; l) \longrightarrow \mathrm{map}(X, Y; l) \xrightarrow{\omega} Y,$$

where ω is the evaluation map at the base point x_0 of X.

We need the following

Lemma 4. *B''_∞ has the same weak homotopy type as $BG_0(F)$.*

In fact, there exists the map $Bi' : BG_0(F) \to BG(F) = B_\infty$ induced by the inclusion $i' : G_0(F) \to G(F)$ (see [2]). Then the map $\pi \circ Bi' : BG_0(F) \to B''_\infty$ induces the isomorphisms of homotopy groups.

Proof of Theorem 1. We have the following commutative diagram

$$\begin{array}{ccccc}
\Omega\,\mathrm{map}_0(B, B'_\infty; k') & \xrightarrow{\Omega(Bj)_\#} & \Omega\,\mathrm{map}_0(B, B_\infty; k) & \xrightarrow{\Omega\pi_\#} & \Omega\,\mathrm{map}_0(B, B''_\infty; \pi\circ k) \\
\downarrow{\scriptstyle \Omega i} & & \downarrow{\scriptstyle \Omega i} & & \downarrow{\scriptstyle \Omega i} \\
\Omega\,\mathrm{map}(B, B'_\infty; k') & \xrightarrow{\Omega(Bj)_\#} & \Omega\,\mathrm{map}(B, B_\infty; k) & \xrightarrow{\Omega\pi_\#} & \Omega\,\mathrm{map}(B, B''_\infty; \pi\circ k) \\
\downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\
F & \xrightarrow{\quad j \quad} & G(F) & \xrightarrow{\quad \Omega\pi \quad} & \Omega B''_\infty
\end{array}$$

Here it should be noticed that $\Omega B''_\infty$ has the same weak homotopy type as $G_0(F)$ by Lemma 4.

Now, we regard the fibration on the left hand of the above diagram as the following fibration:

$$\mathrm{map}_0(B, F) \longrightarrow \mathrm{map}(B, F) \xrightarrow{\omega} F.$$

Thus, by Proposition 3 we see that

$$(\Omega i)_* : \pi_r(\mathrm{map}_0(B, F; l)) \longrightarrow \pi_r(\mathrm{map}(B, F; l))$$

is a monomorphism for every r. On the other hand, we can easily see that the homomorphism

$$(\Omega(Bj)_\#)_* : \pi_r(\Omega\,\mathrm{map}(B, B'_\infty; k')) \longrightarrow \pi_r(\Omega\,\mathrm{map}(B, B_\infty; k))$$

is a monomorphism for every r. In other words, the homomorphism of $\pi_r(\mathcal{G}(E \bmod F))$ into $\pi_r(\mathcal{G}(E))$ induced by the inclusion is a monomorphism for every r. This implies that the following homotopy sequence of the fibration ρ is exact for every $r \geqq 0$

$$1 \to \pi_r(\mathcal{G}(E \bmod F)) \longrightarrow \pi_r(G_0(E)) \longrightarrow \pi_r(R \times G_{0l}(F)) \longrightarrow 1.$$

Correction of the previous paper [5]. On p. 16, line 18, "a map of B to B'" should be replaced by "a map of CW complex B to CW complex B'".

References

[1] G. Allaud: On the classification of fiber spaces. Math. Z., **92**, 110–125 (1966).
[2] J. P. May: Classifying spaces and fibrations. Memoirs Amer. Math. Soc., no. 155 (1975).
[3] Y. Nomura: Homotopy equivalences in a principal fibre space. Math. Z., **92**, 380–388 (1966).
[4] J. Stasheff: *H*-spaces from a homotopy point of view. Lect. Notes in Math., no. 161, Springer-Verlag, Berlin-Heidelberg-New York (1970).
[5] T. Yamanoshita: On the spaces of self homotopy equivalences for fibre spaces. Proc. Japan Acad., **61**A, 15–18 (1985).
[6] ——: On the spaces of self homotopy equivalences for fibre spaces I (preprint).
[7] ——: ditto II (to appear in Publ. RIMS, Kyoto Univ.).