# 85. A Nonsymmetric Partial Difference Functional Equation Analogous to the Wave Equation

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§1. **Introduction.** The purpose of this note is to announce the general solution of the nonsymmetric partial difference functional equation

$$\text{(N)}\qquad \frac{f(x+t,y)+f(x-t,y)-2f(x,y)}{t^2}=\frac{f(x,y+s)+f(x,y-s)-2f(x,y)}{s^2}$$

analogous to the well-known wave equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)f(x,y)=0$$

with the aid of generalized polynomials when no regularity assumptions are imposed on $f$.

Let $R$ be the set of all real numbers, and let $f$ be a function on the plane $R\times R$ taking values in $R$. Define the divided symmetric partial difference operators $\underset{x,t}{\triangle}$ and $\underset{y,t}{\triangle}$ by

$$(\underset{x,t}{\triangle} f)(x,y)=[f(x+t/2,y)-f(x-t/2,y)]/t$$
$$(\underset{y,t}{\triangle} f)(x,y)=[f(x,y+t/2)-f(x,y-t/2)]/t$$

for all $x, y\in R$ and for all $t\in R\backslash\{0\}$.

The symmetric partial difference functional equation

$$((\underset{x,t}{\triangle}{}^2-\underset{y,t}{\triangle}{}^2)f)(x,y)=0$$

analogous to the wave equation or, in expanded form,

$$f(x+t,y)+f(x-t,y)=f(x,y+t)+f(x,y-t)$$

for all $x, y, t\in R$ has been studied by J. Aczél, H. Haruki, M. A. McKiernan and G. N. Sakovič [1], J. A. Baker [2], D. P. Flemming [3], D. Girod [4], H. Haruki [5], M. Kucharzewski [7], M. A. McKiernan [10], and others.

In this note we will consider the nonsymmetric partial difference functional equation

$$((\underset{x,t}{\triangle}{}^2-\underset{y,s}{\triangle}{}^2)f)(x,y)=0$$

which is equivalent to the above expanded form (N) for all $x, y\in R$ and for all $s, t\in R\backslash\{0\}$ and $s\neq t$. Equation (N) is stated in [3] without finding a solution.

§2. **The general solution of (N).** The result is as follows.

**Theorem 1.** *A function $f: R\times R\to R$ satisfies functional equation* (N) *for all $x, y\in R$, $s, t\in R\backslash\{0\}$, and $s\neq t$ if and only if there exist*

(i) *additive functions $A, B: R\to R$,*

(ii) *a function $C: R\times R\to R$ which is additive in both variables, and*

(iii) *polynomials*

$$P_1(x)=a_1+a_3x^2/2+a_5x^3/6$$
$$P_2(y)=a_7+a_3y^2/2+a_4y^3/6$$
$$P_3(x, y)=a_5xy^2/2+a_4yx^2/2+a_6xy^3/6+a_6yx^3/6,$$

*where* $a_1, a_3, a_4, a_5, a_6$, *and* $a_7$ *are constants, such that*

$$f(x, y)=A(x)+B(y)+C(x, y)+P_1(x)+P_2(y)+P_3(x, y)$$

*for all* $x, y \in R$.

If some suitable regularity assumptions are imposed on $f$, then by applying well-known results of an additive function, Theorem 1 implies that $f$ of (N) is given by a certain ordinary polynomial of bounded degree.

The general solution of (N) is obtained by algebraic manupulations.

**§3. A process of the proof.** In order to solve equation (N) we first consider the difference functional equation

$$(\nabla_y^2\psi)(x)=\phi(x)$$

where $\psi, \phi: R \to R$ and the symmetric divided difference operator $\nabla_y$ is defined by

$$(\nabla_y\psi)(x)=[\psi(x+y/2)-\psi(x-y/2)]/y.$$

The above equation is equivalent to

(P) $$[\psi(x+y)+\psi(x-y)-2\psi(x)]/y^2=\phi(x)$$

for all $x \in R$ and $y \in R\backslash\{0\}$. It is clear that if $\phi\equiv 0$, then $\psi$ also satisfies the difference functional equation

$$(\Delta_y^2\psi)(x)=0$$

for all $x, y \in R$. Here $\Delta_y := E^y - I$ is the ordinary forward difference operator,

$$(E^y\psi)(x):=\psi(x+y), \qquad (\Delta_y^{n+1}\psi)(x):=((\Delta_y(\Delta_y^n))\psi)(x)$$

for a given integer $n\geq 1$, and $I$ is the identity operator. Notice that the ring of operators generated by this family of operators is commutative and distributive. It was shown by S. Mazur and W. Orlicz [8] and M. A. McKiernan [9], among others, that the general solution of the finite difference functional equation

(D) $$(\Delta_y^{n+1}\psi)(x)=0$$

for all $x, y \in R$ can be expressed in terms of symmetric multi-additive functions. Specifically, let $A_k$ denote a symmetric function on $R^k \to R$, additive in each variable. Let $A^k$ be the diagonalization of $A_k$, that is, $A^k$ is a map from $R$ to $R$ defined by

$$A^k(x)=A_k(x_1, \cdots, x_k)_{x_1=\cdots=x_k=x}.$$

Then the general solution of (D) is given by a generalized polynomial of degree at most $n$ such that

$$\psi(x)=A^0(x)+A^1(x)+A^2(x)+\cdots+A^n(x)$$

for all $x \in R$, where $A^0(x)=A^0$ is taken to be a constant.

The proof of Theorem 1 is based on the following theorem:

**Theorem 2.** *Two functions* $\psi, \phi: R \to R$ *satisfy equation* (P) *for all* $x \in R$ *and* $y \in R\backslash\{0\}$ *if and only if there exists an additive function* $A^1: R \to R$ *such that*

$$\psi(x)=A^0+A^1(x)+Bx^2/(2!)+Cx^3/(3!)$$
$$\phi(x)=B+Cx$$

*where $A^0$, $B$ and $C$ are constants.*

To prove Theorem 2 we need the following two lemmas. One of them is:

**Lemma 1.** *If two unknown functions $\psi, \phi: R\to R$ satisfy equation* (P) *for all $x\in R$ and $y\in R\backslash\{0\}$, then $\phi$ also satisfies the equation*

$$(\Delta_y^2\phi)(x)=0$$

*for all $x, y\in R$.*

The other is:

**Lemma 2.** *Retain all assumptions of Lemma* 1. *Then $\psi$ also satisfies the equation*

$$(\Delta_y^4\psi)(x)=0$$

*for all $x, y\in R$.*

In addition, some regularity assumptions are imposed only on $\psi$ in the above Theorem 2, then it can be readily shown that $\psi$ and $\phi$ are ordinary polynomials of bounded degree. For example, it is known in [6] that if $\psi$ satisfies equation (D) for all $x, y\in R$ and is bounded on a set of positive Lebesgue measure, then $\psi\in C^\infty$ and the only solution of (D) is given by an ordinary polynomial of degree at most $n$. Hence, we have the following result: Let $\psi$ be bounded on a set of positive Lebesgue measure. Then $\psi, \phi\in C^\infty$ and the only solutions of (P) are given by

$$\psi(x)=c_0+c_1x+c_2x^2/(2!)+c_3x^3/(3!)$$
$$\phi(x)=d^2\psi(x)/dx^2=c_2+c_3x$$

where $\{c_j\}$ are constants. These are also the only continuous solutions of (P).

## References


[1] J. Aczél, H. Haruki, M. A. McKiernan, and G. N. Sakovič: Aequationes Math., **1**, 37–53 (1968).
[2] J. A. Baker: Canad. Math. Bull., **12**, 837–846 (1969).
[3] D. P. Flemming: SIAM J. Appl. Math., **23**, 221–224 (1972).
[4] D. Girod: Aequationes Math., **9**, 157–164 (1973).
[5] H. Haruki: ibid., **5**, 118–119 (1970).
[6] J. H. B. Kemperman: Trans. Amer. Math. Soc., **86**, 28–56 (1957).
[7] M. Kucharzewski: Aequationes Math., **4**, 399–400 (1970).
[8] S. Mazur and W. Orlicz: Studia Math., **5**, 50–68 (1934).
[9] M. A. McKiernan: Ann. Polon. Math., **19**, 331–336 (1967).
[10] ——: Aequationes Math., **8**, 263–266 (1972).