82. *On Multi-products of Pseudo-differential Operators in Gevrey Classes and its Application to Gevrey Hypoellipticity*

By Kazuo TANIGUCHI
Osaka Prefectural University

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§ 1. Introduction and pseudo-differential operators in Gevrey classes. In [6] we give an estimate of multi-products of pseudo-differential operators with symbols in $S^m_{G(\kappa)}$. This corresponds to the case $\rho=1$ and $\delta=0$ in the sense of Hörmander [3]. In the present paper we treat the general case of (ρ, δ). As an application, we improve a result of Gevrey hypoellipticity obtained by Hashimoto-Matsuzawa-Morimoto [2]. The detailed background and description will be published elsewhere.

The symbols we want to treat in this paper are the following:

Definition. Let $m \in \boldsymbol{R}$, $\kappa \geqq 1$, $\kappa' \geqq 1$, $\theta \geqq 0$ and $0 \leqq \delta \leqq \rho \leqq 1$, $\delta < 1$ with $\kappa(1-\delta) \geqq 1$. We say that a symbol $p(x, \xi)$ belongs to a class $SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ if $p(x, \xi)$ satisfies

i) there exist constants C, M and h such that

$$(1) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq CM^{-(|\alpha|+|\beta|)}\alpha!^{\kappa'}(\beta!^{\kappa}+\beta!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta|})\langle\xi\rangle^{m-\rho|\alpha|} \quad \text{if } \langle\xi\rangle \geqq h|\alpha|^{\theta},$$

ii) for any multi-index α there exists a constant C_α such that

$$(2) \qquad |p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_\alpha M^{-|\beta|}(\beta!^{\kappa}+\beta!^{\kappa(1-\delta)}\langle\xi\rangle^{\delta|\beta|})\langle\xi\rangle^{m-\rho|\alpha|}$$

for all x, ξ, where M is a constant independent of α and β.

This definition owes to C. Iwasaki (see also [4]). We also note that the class $SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ contains the class $S^m_{\rho,\delta,\sigma}$ studied in [2] if we set $\kappa=\sigma/(\rho-\delta)$, $\kappa'=1$ and $\theta=\sigma/(\rho-\delta)$ (=their θ).

Let $P=p(X, D_x)$ denote a pseudo-differential operator with a symbol $p(x, \xi) \in SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ defined by

$$Pu=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi, \qquad u \in \mathcal{S},$$

where $\hat{u}(\xi)$ is a Fourier transform of u. Then, by the method of [7] we can prove

Proposition 1. *Let $\mathcal{D}^{\{\kappa\}\prime}_{L^2}$ be a class of ultradistributions studied in* [7]. *Then, pseudo-differential operators with symbols in $SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ act on $\mathcal{D}^{\{\kappa\}\prime}_{L^2}$ and their images are also contained in $\mathcal{D}^{\{\kappa\}\prime}_{L^2}$.*

This proposition was first proved by the author in the case of $\rho=1$, $\delta=0$ and by C. Iwasaki in the case of $\delta>0$.

Proposition 2. *Let $\kappa>1$ and let* $\mathrm{WF}_{G(\kappa)}(u)$ *be the wave front set of u in the Gevrey class of order κ. Assume $\rho>0$ and $\tilde{\kappa} \geqq \max(\kappa, \theta, \kappa'/\rho)$. Then for $p(x, \xi) \in SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ we have*

$$\mathrm{WF}_{G(\kappa)}(Pu) \subset \mathrm{WF}_{G(\kappa)}(u).$$

§2. **Multi-products.** Let $P_j = p_j(X, D_x)$, $p_j \in SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ with $m \geqq 0$. Consider

(3) $$Q_{\nu+1} = P_1 P_2 \cdots P_{\nu+1}.$$

Theorem 1. *Assume that each $p_j(x, \xi)$ satisfies* (1) *and* (2) *with constants C, M, h and C_α independent of j. Denote $\tilde{C} = \max_{|\alpha| \leqq n_0}(C, C_\alpha)$, where $n_0 = 2[n/2+1]$. We assume $\theta = \kappa$ if $\kappa' = \rho = 1$, or $\theta > 0$ if $\rho < 1$ and $\kappa' = 1$. Then, the symbols $q_{\nu+1}(x, \xi)$ of* (3) *are represented as*

$$q_{\nu+1}(x, \xi) = q^0_{\nu+1}(x, \xi) + \tilde{q}_{\nu+1}(x, \xi)$$

and $q^0_{\nu+1}(x, \xi)$ and $\tilde{q}_{\nu+1}(x, \xi)$ satisfy

(4) $$|q^{0}{}^{(\alpha)}_{\nu+1(\beta)}| \leqq C^{\nu+1} A^\nu M_1^{-(|\alpha|+|\beta|)} \alpha!^{\kappa'} (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle\xi\rangle^{\delta|\beta|}) \langle\xi\rangle^{(\nu+1)m - \rho|\alpha|}$$
$$\text{if } \langle\xi\rangle \geqq h_1 |\alpha|^\theta,$$

(5) $$|q^{0}{}^{(\alpha)}_{\nu+1(\beta)}| \leqq \tilde{C}^{\nu+1} A^\nu C'_\alpha M_1^{-|\beta|} (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle\xi\rangle^{\delta|\beta|}) \langle\xi\rangle^{(\nu+1)m - \rho|\alpha|},$$

(6) $$|\tilde{q}^{(\alpha)}_{\nu+1(\beta)}| \leqq \tilde{C}^{\nu+1} A^\nu C^\nu_m C'_\alpha M_1^{-|\beta|} [(\nu+1)m]!^\kappa \beta!^\kappa \exp(-\varepsilon \langle\xi\rangle^{1/\kappa}).$$

In (4)–(6) *the constants A, M_1, C'_α, C_m, h_1 and ε (>0) are independent of ν and β.*

The proof will be done by using the inductive method of ν and the idea used in §1 of [5], where we divide the symbols $p(x, \xi)$ of Fourier integral operators into the sum of symbols $p^0(x, \xi; \zeta)$ and $\tilde{p}(x, \xi; \zeta)$. These symbols depend also on a parameter ζ and satisfy "$|\xi - \zeta| \leqq \langle\zeta\rangle/8$ on supp p^0 and $|\zeta - \xi| \geqq \langle\zeta\rangle/10$ on supp $\tilde{p}$". We make this division by using cut functions in Gevrey classes.

Corollary. *Let $p(x, \xi) \in SG^0_{\rho,\delta;\kappa,\kappa',\theta}$ satisfy* (1) *and* (2) *with $C < A$ and $C_\alpha < A$ ($|\alpha| \leqq n_0$) for A in the above theorem. Then the inverse operator of $I - P$ is obtaind by the Neumann series $\sum_{\nu=0}^{\infty} P^\nu$ and it is represented as the sum $r^0(X, D_x) + \tilde{r}(X, D_x)$ of pseudo-differential operators $r^0(X, D_x)$ and $\tilde{r}(X, D_x)$ with symbols $r^0(x, \xi) \in SG^0_{\rho,\delta;\kappa,\kappa',\theta}$ and $r(x, \xi) \in \mathcal{R}_{G(\kappa)}$, where $\mathcal{R}_{G(\kappa)}$ is a class of regularizers defined in Definition* (S) *of* [6].

§3. **Gevrey hypoellipticity.** Let $P = p(X, D_x)$ be a differential operator with coefficients in a Gevrey class $\gamma^{(\sigma)}$ of order σ. Assume

(7) $$|p(x, \xi)| \geqq C \langle\xi\rangle^{m'} \quad \text{for large } |\xi|,$$

(8) $$|p^{(\alpha)}_{(\beta)}(x, \xi)/p(x, \xi)| \geqq CM^{-(|\alpha|+|\beta|)} \alpha!\, \beta!^\sigma \langle\xi\rangle^{-\rho|\alpha| + \delta|\beta|}$$
$$\text{for } \langle\xi\rangle \geqq h|\alpha|^\theta \ (\theta = \sigma/(\rho - \delta))$$

and $\rho > \delta$. Under these conditions Hashimoto-Matsuzawa-Morimoto [2] constructed a parametrix $Q = q(X, D_x)$ of P as $q(x, \xi) \in SG^{-m'}_{\rho,\delta;\theta,1,\theta}$ ($\theta = \sigma/(\rho - \delta)$) and $QP - I$ is an integral operator with a kernel in the Gevrey class of order $\theta = \sigma/(\rho - \delta)$. Set $\kappa = \sigma/(1 - \delta)$. Then, we can generalize (8) as

(8)′ $$|p^{(\alpha)}_{(\beta)}(x, \xi)/p(x, \xi)| \leqq CM^{-(|\alpha|+|\beta|)} \alpha!^{\kappa'} (\beta!^\kappa + \beta!^{\kappa(1-\delta)} \langle\xi\rangle^{\delta|\beta|}) \langle\xi\rangle^{-\rho|\alpha|}$$
$$\text{for } \langle\xi\rangle \geqq h|\alpha|^\theta.$$

Now, consider a pseudo-differential operator $P = p(X, D_x)$ with a symbol $p(x, \xi) \in SG^m_{\rho,\delta;\kappa,\kappa',\theta}$ and assume (7), (8)′ and $\rho > \delta$. Then, by using Corollary in §2 we can prove

Theorem 2. *The parametrix Q of $P = p(X, D_x)$ is constructed as $Q = q(X, D_x)$, $q(x, \xi) \in SG^{-m'}_{\rho,\delta;\kappa,\kappa',\theta}$ and it satisfies*

$$QP-I \in \mathcal{R}_{G(\kappa)}.$$

For example, consider an operator

$$P=x_2^4(iD_{x_1}+D_{x_2}^2)+1 \qquad \text{in } R_x^2$$

following T. Matsuzawa. Then, modifying x_2^4 for large $|x_2|$ we can prove that its symbol satisfies (7) and (8)′ with $\kappa=4/3$ and $\rho=1/2$, $\delta=1/4$. So, by our method we can construct its parametrix Q as $Q=q(X, D_x)$, $q(x, \xi) \in SG^0_{1/2,1/4;4/3,1,4/3}$ such that $QP-I \in \mathcal{R}_{G(4/3)}$. So, we have

$$\mathrm{WF}_{G(2)}(Pu)=\mathrm{WF}_{G(2)}(u).$$

This result is an improvement of the one in [2], since we obtain only $\mathrm{WF}_{G(4)}(Pu)=\mathrm{WF}_{G(4)}(u)$ by their method.

As another example, we consider an operator

$$P=D_{x_1}^2+a_k(x_1)D_{x_2}^2+D_{x_3}^2 \qquad \text{in } R_x^3,$$

where $a_k(t)$ belongs to $\gamma^{((k+1)/k)}(R_t^1)$ and satisfies $a_k(t)=t^{2k}$ ($|t|\leqq 1$) and $|a_k(t)| \leqq 1$ ($|t|\leqq 2$). Then, we can construct a parametrix Q of P as $Q=q(X, D_x)$, $q(x, \xi) \in SG^{-2\delta}_{\delta,\delta;\kappa,1,\kappa}$ ($\delta=1/(k+1)$, $\kappa=(k+1)/k$ $[=1/(1-\delta)]$) and $QP-I \in \mathcal{R}_{G(\kappa)}$. So, we obtain

(9) $$u \in \gamma^{((k+1)/k)}(R_x^3) \qquad \text{if } Pu \in \gamma^{((k+1)/k)}(R_x^3)$$

and

(10) $$\mathrm{WF}_{G(k+1)}(Pu)=\mathrm{WF}_{G(k+1)}(u).$$

We note that by Baouendi-Goulaouic [1] (the case $k=1$) and Y. Morimoto (the general case) the property (10) is optimal for the index κ of the estimation $\mathrm{WF}_{G(\kappa)}(Pu)=\mathrm{WF}_{G(\kappa)}(u)$.

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