# 80. *Removable Singularities of Real Analytic Solutions of Fuchsian Partial Differential Equations*

By Toshinori ÔAKU
Department of Mathematics, University of Tokyo

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A. Kaneko has studied continuation problem of real analytic solutions of linear partial differential equations systematically by using the theory of non-characteristic boundary value problem for hyperfunctions (see, e.g. expository papers [3] and [4]). As for equations with analytic coefficients, he has proved fundamental results for solutions having singularities (i.e. points where the solution is not defined) contained in a real analytic hypersurface which is non-characteristic for the equation ([2]). Here we extend results in [2] to the case where the equation is of Fuchsian type with respect to the hypersurface. Our main tool is the theory of micro-hyperbolic boundary value problems developed in Ôaku [8], [9].

Let $M$ be an open subset of $\boldsymbol{R}^n$ and $N$ be a real analytic hypersurface in $M$. Since the problems considered in this paper are of local character, we may assume that $N=\{x=(x_1, x')\in M;\ x_1=0\}$ with the notation $x'=(x_2, \cdots, x_n)$. We use the notation $D'=(D_2, \cdots, D_n)$ with $D_j=\partial/\partial x_j$. We put $M_\pm=\{x\in M;\ \pm x_1>0\}$ and set $\mathcal{B}_{N|M_\pm}=(\iota_\pm)_*(\iota_\pm)^{-1}\mathcal{B}_M|_N$, where $\iota_\pm : M_\pm \to M$ are the embeddings and $\mathcal{B}_M$ is the sheaf of hyperfunctions on $M$. Hence sections of $\mathcal{B}_{N|M_\pm}$ are hyperfunctions on the intersection of $M_\pm$ and of a neighborhood of a point of $N$.

We assume that a linear partial differential operator $P$ with real analytic coefficients is a Fuchsian operator of weight $m-k$ with respect to $N$ in the sense of Baouendi-Goulaouic [1]: $P$ is written in the form

$$P=a(x)(x_1^k D_1^m + A_1(x, D')x_1^{k-1}D_1^{m-1}+\cdots+A_k(x, D')D_1^{m-k}+\cdots+A_m(x, D'));$$

here $a(x)$ is a non-vanishing real analytic function, $k$ and $m$ are integers with $0\leqq k\leqq m$, $A_j(x, D')$ is an operator of order $\leqq j$ for $1\leqq j\leqq m$, and $A_j(0, x', D')$ is of order 0, i.e. equals a real analytic function $a_j(x')$, for $1\leqq j\leqq k$. The roots $\lambda=0, 1, \cdots, m-k-1, \lambda_1(x'), \cdots, \lambda_k(x')$ of the equation

$$\lambda(\lambda-1)\cdots(\lambda-m+1)+a_1(x')\lambda(\lambda-1)\cdots(\lambda-m+2)$$
$$+\cdots+a_k(x')\lambda(\lambda-1)\cdots(\lambda-m+k+1)=0$$

are called the characteristic exponents of $P$. For a point $\mathring{x}=(0, \mathring{x}')$ of $N$, we define a condition $C(\mathring{x})$ by

$$C(\mathring{x}) : \lambda_i(\mathring{x}')\notin \boldsymbol{Z},\quad \lambda_i(\mathring{x}')-\lambda_j(\mathring{x}')\notin \boldsymbol{Z}\backslash\{0\}\qquad \text{for any } 1\leqq i,\ j\leqq k.$$

We set $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_X P$, where $\mathcal{D}_X$ denotes the sheaf of linear partial differential operators with holomorphic coefficients on a complex neighborhood $X$ of $M$. Then $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+})$ is the sheaf of $\mathcal{B}_{N|M_+}$-solutions of $\mathcal{M}$.

**Proposition 1.** *Assume $C(\mathring{x})$ for any $\mathring{x}\in N$. Then there exists an* *injective homomorphism*

$$\gamma_+ : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow (\mathcal{B}_N)^m.$$

*$\gamma_+$ is decomposed in the form $\gamma_+(u)=(\gamma_{+\mathrm{reg}}(u), \gamma_{+\mathrm{sing}}(u))$ with $\gamma_{+\mathrm{reg}}(u) \in (\mathcal{B}_N)^{m-k}$ and $\gamma_{+\mathrm{sing}}(u) \in (\mathcal{B}_N)^k$ for $u \in \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+})$.*

Sketch of the proof. For a $\mathcal{B}_{N|M_+}$-solution $u$ of $\mathcal{M}$, put $v={}^t(u_0, u_1, \cdots, u_{m-1})$ with $u_0=u$, $u_1=D_1u, \cdots, u_{m-k}=D_1^{m-k}u$, $u_{m-k+1}=(x_1D_1)D_1^{m-k}u, \cdots, u_{m-1}=(x_1D_1)^{k-1}D_1^{m-k}u$. Using Theorem 1.3.6 of Tahara [10], we can show that there exists an invertible matrix $R$ of $\mathcal{O}_0\tilde{\mathcal{D}}$ (cf. [7]) such that $w=R^{-1}v$ satisfies an equation

$$x_1D_1w=\begin{pmatrix} -I_{m-k} & 0 \\ 0 & A''(x') \end{pmatrix}w;$$

here $I_{m-k}$ is the identity matrix of degree $m-k$, $A''(x')$ is a $k\times k$ matrix of analytic functions on $N$ which does not have any integer as an eigenvalue. Hence $w$ is written in the form

$$w=\begin{pmatrix} x_1^{-1}f'(x') \\ x_1^{A''(x')}f''(x') \end{pmatrix}$$

with $f' \in (\mathcal{B}_N)^{m-k}$ and $f'' \in (\mathcal{B}_N)^k$. We put $f'=\gamma_{+\mathrm{reg}}(u)$ and $f''=\gamma_{+\mathrm{sing}}(u)$. The injectivity of $\gamma_+$ is proved in the same way as the proof of Theorem 2 of [7].

We denote by $\mathcal{B}^F_{N|M_+}$ the subsheaf of $\mathcal{B}_{N|M_+}$ consisting of F-mild hyperfunctions from the positive side of $N$ (cf. [5] and [7]).

**Proposition 2.** *Under the same assumptions as Proposition 1, let $u$ be a $\mathcal{B}_{N|M_+}$-solution of $\mathcal{M}$. Then $u$ is* F*-mild if and only if $\gamma_{+\mathrm{sing}}(u)=0$.*

Changing the sign of $x_1$, we get an injective homomorphism

$$\gamma_- : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_-}) \longrightarrow (\mathcal{B}_N)^m$$

with $\gamma_-=(\gamma_{-\mathrm{reg}}, \gamma_{-\mathrm{sing}})$ under the same assumptions as in Proposition 1.

**Proposition 3.** *Under the condition $C(\mathring{x})$ for any $\mathring{x} \in N$, let $u_\pm$ be $\mathcal{B}_{N|M_\pm}$-solutions of $\mathcal{M}$. Then there exists a hyperfunction solution $u$ of $\mathcal{M}$ on a neighborhood of $N$* (i.e. *a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)|_N$*) *such that $u|_{M_\pm}=u_\pm$ if and only if $\gamma_{+\mathrm{reg}}(u_+)=\gamma_{-\mathrm{reg}}(u_-)$. Moreover such $u$ is unique. If $\gamma_{+\mathrm{sing}}(u_+)=\gamma_{-\mathrm{sing}}(u_-)=0$ in addition to the above assumption, then $u$ has $x_1$ as a real analytic parameter.*

We remark that the last assertion of Proposition 3 follows from Proposition 2.

We define closed subsets $V^\pm_{N,A}(P)$ ($A$-boundary characteristic points of $P$) of $S^*_NY=\sqrt{-1}S^*N$ as follows:

**Definition.** A point $x^*=(\mathring{x}', \sqrt{-1}\mathring{\xi}'\infty)$ of $\sqrt{-1}S^*N$ with $\mathring{x}' \in \boldsymbol{R}^{n-1}$ and $\mathring{\xi}' \in S^{n-2}$ is not contained in $V^\pm_{N,A}(P)$ if and only if there exists $\varepsilon>0$ such that $\sigma(P)(x, \zeta_1, \sqrt{-1}\xi')\neq 0$ for any $x \in \boldsymbol{R}^n$ with $0<\pm x_1<\varepsilon$, $|x'-\mathring{x}'|<\varepsilon$, for any $\xi' \in \boldsymbol{R}^{n-1}$ with $|\xi'-\mathring{\xi}'|<\varepsilon$, and for any $\zeta_1 \in \boldsymbol{C}$ with $\pm\mathrm{Re}\,\zeta_1<0$; here $\sigma(P)$ denotes the principal symbol of $P$. We put

$$V_{N,A}(P)=V^+_{N,A}(P)\cup V^-_{N,A}(P).$$

This is a generalization of the definition by Kaneko [2] in the non-characteristic case. By Theorem 1 and Lemma of [8], we get

**Proposition 4.** *Under the same assumption as in Proposition* 1, *let* $u_{\pm}$ *be a real analytic solution of* $\mathcal{M}$ *on* $M_{\pm}$. *Then the singular spectrum of* $\gamma_{\pm}(u_{\pm})$ *is contained in* $V^{\pm}_{N,A}(P)$.

Now we can generalize Theorem 3.1 of [2]:

**Theorem 1.** *Let* $\mathring{x}$ *be a point of* $N$ *and let* $\varphi$ *be a real valued* $C^1$ *function on* $N$ *such that* $\varphi(\mathring{x})=0$ *and* $d\varphi(\mathring{x})\neq 0$. *Let* $K$ *be a closed subset of* $N$ *such that* $\varphi\leqq 0$ *on* $K$. *Assume* $C(\mathring{x})$ *and that* $V_{N,A}(P)$ *does not contain both of the points* $(\mathring{x}, \pm\sqrt{-1}d\varphi(\mathring{x}))\in\sqrt{-1}S^*N$. *Then any real analytic solution* $u$ *of* $\mathcal{M}$ *defined on* $U\backslash K$, *where* $U$ *is a neighborhood of* $\mathring{x}$ *in* $M$, *is uniquely continued as a hyperfunction solution* $\tilde{u}$ *of* $\mathcal{M}$ *to a neighborhood of* $\mathring{x}$ *in* $M$. *Moreover* $\tilde{u}$ *has* $x_1$ *as a real analytic parameter on a neighborhood of* $\mathring{x}$.

This theorem follows from Propositions 3 and 4 by the same argument as in [2].

Under some additional conditions, we can continue $u$ as a real analytic function; we can generalize Theorem I of Kaneko [3]:

**Theorem 2.** *Let* $P$ *be a Fuchsian operator of weight* $m-k$ *with respect to* $N$ *and* $\mathring{x}$ *be a point of* $N$. *Assume* $C(\mathring{x})$ *and*

(i) *For* $x\in M$ *and* $\xi\in\boldsymbol{R}^n$ *the principal symbol of* $P$ *is written in the form* $\sigma(P)(x,\xi)=x_1^k p(x,\xi)$ *with a real valued real analytic function* $p$ *(then* $p$ *is a polynomial of degree* $m$ *in* $\xi_1$*).*

(ii) $\mathrm{grad}_{\xi}\, p\neq 0$ *at* $(\mathring{x},\xi)$ *if* $p(\mathring{x},\xi)=0$ and $\xi\neq 0$;

(iii) *There exists* $\xi'\in\boldsymbol{R}^{n-1}$ *such that the equation* $p(\mathring{x},\zeta_1,\xi')=0$ *in* $\zeta_1$ *has* $m$ *distinct real roots.*

*Under these assumptions, for any open neighborhood* $U$ *of* $\mathring{x}$ *in* $M$, *any real analytic solution of* $Pu=0$ *on* $U\backslash\{\mathring{x}\}$ *is uniquely continued to* $U$ *as a real analytic solution.*

To prove this theorem, we first continue $u$ as a hyperfunction solution $\tilde{u}$ on $U$ by using Theorem 1. Since $\tilde{u}$ has $x_1$ as a real analytic parameter, we can show that micro-analyticity propagates from outside of $\mathring{x}$.

For example, Theorem 2 applies to the operator

$$P=x_1(D_1^2+\cdots+D_k^2-D_{k+1}^2-\cdots-D_n^2)+\sum_{j=1}^{n}a_j(x)D_j+b(x),$$

where $1\leqq k<n$, $a_j$ are real analytic with $a_1(\mathring{x})\notin\boldsymbol{Z}$. Detailed arguments of these results will appear elsewhere.

## References

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