8. *Propagation of Wave Front Sets of Solutions of the Cauchy Problem for a Hyperbolic System in Gevrey Classes*

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Introduction and main theorem. Consider a hyperbolic system

$$(1)\qquad \mathcal{L}=D_t-\begin{pmatrix}\lambda_1(t,X,D_x) & & 0\\ & \ddots & \\ 0 & & \lambda_l(t,X,D_x)\end{pmatrix}+(b_{jk}(t,X,D_x))$$

on $[0,T]\times R^n_x$

with real symbols $\lambda_j(t,x,\xi)$ in $G^{(\kappa)}([0,T];S^1_{G(\kappa)})$ and symbols $b_{jk}(t,x,\xi)$ in $G^{(\kappa)}([0,T];S^\sigma_{G(\kappa)})$ $(0\leqq\sigma<1/\kappa)$. Here, for $\kappa>1$ we denote by $G^{(\kappa)}([0,T];S^m_{G(\kappa)})$ a class of symbols $p(t,x,\xi)$ of pseudo-differential operators satisfying

$$|\partial_t^\gamma\partial_\xi^\alpha\partial_x^\beta p(t,x,\xi)|\leqq CM^{-(\gamma+|\alpha|+|\beta|)}\gamma!^\kappa\alpha!^\kappa\beta!^\kappa\langle\xi\rangle^{m-|\alpha|}$$

for constants C and M. In the recent paper [9] the second author has constructed the fundamental solution of (1) assuming the constant multiplicities of characteristic roots of $\mathcal{L}$ and investigated the propagation of wave front sets for the solution of the Cauchy problem of $\mathcal{L}$:

$$(2)\qquad \mathcal{L}U(t)=0\quad(0<t\leqq T_0),\quad U(0)=G.$$

In the present paper we study the propagation of wave front sets in Gevrey classes for the solution $U(t)$ of (2) without assuming the constant multiplicity and get a similar result to the one for the C^∞ case obtained by Kumano-go and the second author [4].

Let $\varepsilon>0$ and let V be a conic set in $T^*(R^n_x)$. Then, we denote by $\Gamma^\nu_\varepsilon(t,V)$ $(\nu=0,1,\cdots)$ the set of end points (at t) of all ε-admissible trajectories of, at most, step ν issuing from the ε-conic neighborhood $V_\varepsilon\equiv\{(x,\xi);|x-y|\leqq\varepsilon,$ $|\xi/|\xi|-\eta/|\eta||\leqq\varepsilon,(y,\eta)\in V\}$ of V (concerning the characteristic roots $\lambda_j(t,x,\xi)$, $j=1,\cdots,l$; cf. [2]) and set

$$(3)\qquad \begin{cases}\Gamma_\varepsilon(t,V)=\text{the closure of }\bigcup_{\nu=0}^\infty\Gamma^\nu_\varepsilon(t,V),\\ \Gamma(t,V)=\bigcap_{\varepsilon>0}\Gamma_\varepsilon(t,V).\end{cases}$$

We also denote by $\mathcal{D}^{\{\kappa\}\prime}_{L^2}$ a class of ultradistributions defined in [3] (see also [11]).

Theorem. *Let* $\mathcal{L}$ *be a hyperbolic operator of the form* (1) *with* $\lambda_j(t,x,\xi)\in G^{(\kappa)}([0,T];S^1_{G(\kappa)})$ *and* $b_{jk}(t,x,\xi)\in G^{(\kappa)}([0,T];S^\sigma_{G(\kappa)})$ *for* $0\leqq\sigma<1/\kappa$. *Consider the Cauchy problem* (2). *Then, there exists a unique solution* $U(t)$ *in* $\mathcal{B}^1([0,T_0];\mathcal{D}^{\{\kappa\}\prime}_{L^2})$ $(0<T_0\leqq T)$ *for any* $G\in\mathcal{D}^{\{\kappa\}\prime}_{L^2}$ *and it satisfies*

$$(4)\qquad \mathrm{WF}_{G(\kappa_1)}(U(t))\subset\Gamma(t,\mathrm{WF}_{G(\kappa_1)}(G))$$

for any κ_1 *satisfying* $\kappa\leqq\kappa_1<1/\sigma$.

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Here, for $\kappa_1 \geqq \kappa$, $\mathrm{WF}_{G(\kappa_1)}(u)$ is a wave front set of $u \in \mathcal{D}_{L^2}^{\{\kappa\}'}$ defined as follows:

Definition (cf. [11]). Let $u \in \mathcal{D}_{L^2}^{\{\kappa\}'}$ and $\kappa_1 \geqq \kappa$. Then, the point (x_o, ξ_o) in $T^*(R_x^n)$ does not belong to $\mathrm{WF}_{G(\kappa_1)}(u)$ if there exists a symbol $a(x, \xi)$ in $S^0_{G(\kappa)}$ with $a(x_o, \theta\xi_o) \neq 0$ $(\theta \geqq 1)$ such that $f(x) = a(X, D_x)u$ satisfies

$$|\partial_x^\alpha f(x)| \leqq CM^{-|\alpha|}\alpha!^{\kappa_1} \qquad \text{for all } x \in R_x^n.$$

We remark that this definition is equivalent to the definition given by Hörmander [1]. We also remark that the first author studies the best possibility of (4) in [6].

§ 1. Hyperbolic differential operators. Consider the Cauchy problem

$$(1.1) \qquad Lu = 0 \quad (0 < t \leqq T_o), \quad \partial_t^j u(0) = g_j \quad (j = 0, 1, \cdots, m-1)$$

for a hyperbolic operator

$$(1.2) \qquad L = D_t^m + \textstyle\sum_{j=0}^{m-1} \sum_{|\alpha| \leqq m-j} a_{j,\alpha}(t, x) D_x^\alpha D_t^j \qquad \text{on } [0, T] \times R_x^n$$

with coefficients $a_{j,\alpha}(t, x)$ in a Gevrey class $\gamma^{(\kappa)}([0, T] \times R_x^n)$, that is, they satisfy

$$|\partial_t^\gamma \partial_x^\beta a_{j,\alpha}(t, x)| \leqq CM^{-(\gamma + |\beta|)}\gamma!^\kappa \beta!^\kappa \qquad \text{for } (t, x) \in [0, T] \times R_x^n.$$

In [9] we have shown that the problem (1.1) is reduced to the equivalent Cauchy problem (2) with $\sigma = (r-q)/r$ under the condition that there exist regularly hyperbolic operators $L_1, L_2, \cdots, L_r$ with coefficients in $\gamma^{(\kappa)}([0, T] \times R_x^n)$ such that L has a form

$$(1.3) \qquad L = L_1 L_2 \cdots L_r + \textstyle\sum_{j=0}^{m-q} \sum_{|\alpha| \leqq m-q-j} a'_{j,\alpha}(t, x) D_x^\alpha D_t^j$$

with $a'_{j,\alpha}(t, x)$ in $\gamma^{(\kappa)}([0, T] \times R_x^n)$ and $1 \leqq q \leqq r$ (see also [5]). So, using Theorem we get

$$(1.4) \qquad \mathrm{WF}_{G(\kappa_1)}(u(t)) \subset \Gamma(t, \textstyle\bigcup_{j=0}^{m-1} \mathrm{WF}_{G(\kappa_1)}(g_j)) \qquad \text{for } \kappa \leqq \kappa_1 < r/(r-q).$$

In the case of $q=1$, that is, in the case of assuming no conditions on lower order terms, Wakabayashi [12] has also investigated the propagation of wave front sets for solutions of (1.1) in the Gevrey class of order $\kappa_1 (\kappa \leqq \kappa_1 < r/(r-1))$ by constructing a parametrix of L and introducing "flows" K_z^+ in $T^*(R_t^1 \times R_x^n)$ emanating from a point z in $T^*(R_t^1 \times R_x^n)$. His result (for the operator (1.3)) is the same as our estimate (1.4), since we have

$$\pi(K_{z_o}^+ \cap \{t = t_o\}) = \Gamma(t_o, \{(x_o, \theta\xi_o); \theta > 0\})$$

for $t_o > 0$ and $z_o \in \pi^{-1}(\{(x_o, \xi_o)\}) \cap \{t=0\} \cap p^{-1}(0)$, where $p = p(t, x, \tau, \xi)$ is the principal symbol of L and $\pi: T^*(R_t^1 \times R_x^n) \to T^*(R_x^n)$ is a projection (cf. Theorem 4.4 in [13]).

As another condition under which the problem (1.1) can be reduced to the problem (2), we consider an operator L of the form

$$(1.5) \qquad L = L_1 L_2 L_3 + P_1 L_1 + P_2 L_2 + P_3 L_3 + P_4.$$

Here, L_j, $j = 1, 2, 3$, are regularly hyperbolic operators of order m_j $(m_1 + m_2 + m_3 = m)$ and P_1, P_2, P_3 and P_4 are differential operators of order, at most, $m - m_1 - 1$, $m - m_2 - 1$, $m - m_3 - 1$ and $m - 1$, respectively, with coefficients in $\gamma^{(\kappa)}([0, T] \times R_x^n)$. We note that if $P_1 = P_2 = P_3 = 0$ then (1.5) is the form (1.3).

Proposition 1. *Let L be a hyperbolic operator of the form* (1.5). *Then, the Cauchy problem* (1.1) *can be reduced to the equivalent Cauchy* *problem* (2) *for an operator* $\mathcal{L}$ *of the form* (1) *with* σ *satisfying the following*:

i) $\sigma=0$ *if order* $P_1 \leqq m-m_1-2$, *order* $P_2 \leqq m-m_2-3$, *order* $P_3 \leqq m-m_3-2$ *and order* $P_4 \leqq m-3$,

ii) $\sigma=1/3$ *if order* $P_j \leqq m-m_j-2$ $(j=1,2,3)$ *and order* $P_4 \leqq m-2$,

iii) $\sigma=1/2$ *if order* $P_j \leqq m-m_j-1$ $(j=1,2,3)$ *and order* $P_4 \leqq m-2$,

iv) $\sigma=2/3$ *otherwise*.

We remark that the case i) is treated in [8]. As shown in this proposition it seems to be very difficult to find the conditions on lower order terms of a hyperbolic operator (1.2) with smooth characteristic roots under which the problem (1.1) is reduced to an equivalent problem (2) of a hyperbolic system (1) with a given σ (<1).

§2. **Proof of Theorem.** Let $\phi_j(t,s;x,\xi)$ be the phase function corresponding to $\lambda_j(t,x,\xi)$ $(j=1,2,\cdots,l)$. Then, as in the C^∞ case ([4], pp. 185–186) the fundamental solution $E(t,s)$ of (1) is constructed in the form

$$E(t,s)=\sum_{j=1}^{l} I_{j,\phi_j}(t,s)+\sum_{\nu=1}^{\infty}\sum_{\substack{j_k=1,\cdots,l\\(k=1,\cdots,\nu+1)}}\int_s^t\int_s^{t_1}\cdots\int_s^{t_{\nu-1}} I_{j_1,\phi_{j_1}}(t,t_1) \times W_{j_2,\phi_{j_2}}(t_1,t_2)\cdots W_{j_{\nu+1},\phi_{j_{\nu+1}}}(t_\nu,s)dt_\nu\cdots dt_1 \quad (t_0=t) \tag{2.1}$$

for $0\leqq t\leqq T_o$

for some T_o $(\leqq T)$, where $I_{j,\phi_j}(t,s)$ is a matrix of Fourier integral operators with phase function $\phi_j(t,s;x,\xi)$ and with symbol 1 ((j,j) element) or 0 (others), and $W_{j,\phi_j}(t,s)$ is the one with symbol $w_j(t,s;x,\xi)$ satisfying

$$|\partial_t^{\gamma}\partial_s^{\gamma'}\partial_\xi^{\alpha}\partial_x^{\beta}w_j(t,s;x,\xi)|\leqq CM^{-(\gamma+\gamma'+|\alpha|+|\beta|)}\gamma!^{\kappa}\gamma'!^{\kappa}\alpha!^{\kappa}\beta!^{\kappa}\langle\xi\rangle^{\sigma-|\alpha|}.$$

Since we assume $\sigma\kappa<1$, the first part of Theorem is verified easily by the results in [11]. For the proof of the inclusion (4) we employ

Proposition 2. *Let* V *be a closed conic set in* $T^*(R_x^n)$ *and let* $\Gamma_\varepsilon(t,V)$ *be a set defined in* (3) *for* $\varepsilon>0$. *Let* $a(x,\xi)$ *and* $b(x,\xi)$ *be symbols in* $S^0_{G(\kappa)}$ *satisfying*

$$\begin{cases}\text{supp}\, b\subset V_{\varepsilon/2},\\ |x-y|\geqq\varepsilon/2 \quad or \quad |\xi/|\xi|-\eta/|\eta||\geqq\varepsilon/2\end{cases} \tag{2.2}$$

if $(x,\xi)\in\text{supp}\, a$ *and* $(y,\eta)\in\Gamma_\varepsilon(t,V)$.

Then, for the fundamental solution $E(t,s)$ *of* (2.1) *the operator* $a(X,D_x)\cdot E(t,0)b(X,D_x)$ *is a pseudo-differential operator with symbol* $p(t,x,\xi)$ *satisfying for some constants* M *and* $\delta>0$

$$|\partial_t^{\gamma}\partial_\xi^{\alpha}\partial_x^{\beta}p(t,x,\xi)|\leqq C_{\alpha,\gamma}M^{-|\beta|}\beta!^{\kappa}e^{-\delta\langle\xi\rangle^{1/\kappa}}$$

with constants $C_{\alpha,\gamma}$ *independent of* β.

Then, we can get (4) as shown in [10]. So, the key point of the proof of (4) is to obtain Proposition 2, which is proved in Morimoto-Taniguchi [7] by the method of the oscillatory integrals.

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