75. Dynkin Graphs and Combinations of Singularities on Quartic Surfaces

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In this article we show that possible combinations of singularities on quartic surfaces in the three-dimensional projective space $\boldsymbol{P}^3$ can be described systematically by Dynkin graphs. Details of the proof will appear elsewhere. We assume that every variety is algebraic and is defined over the complex number field $\boldsymbol{C}$.

Definition 1. A disjoint finite union of connected Dynkin graphs of type A, B, D or E is called a Dynkin graph. For a Dynkin graph, the following procedure is called an *elementary transformation* of it.

(1) Replace each component by the extended Dynkin graph of the corresponding type.

(2) Choose in an arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with the edges issuing from them (cf. Bourbaki [1]).

Note that any connected Dynkin graph of type A, D, or E corresponds to a singularity on a surface (cf. Durfee [2]).

Theorem 2. *Let* $G=\sum_{k\geq 1} a_k A_k+\sum_{l\geq 4} b_l D_l+\sum_{m=6}^{8} c_m E_m$ *(a finite sum) be a Dynkin graph with only components of type A, D or E. Set* $r=\sum a_k k+\sum b_l l+\sum c_m m$. *Then the following conditions (A) and (B) are equivalent.*

(A) There exists a quartic surface in the projective space of dimension 3 whose combination of singularities just agrees with G and moreover one of the following conditions $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$ *holds for the root lattice* $Q=Q(G)$ *of type G.*

$\langle 1\rangle$ $r=17$, *the discriminant* $d(Q)$ *of Q is a square number, and for every prime number p,* $\varepsilon_p(Q)=1$.

$\langle 2\rangle$ $r=16$, *and for every prime number p,* $\varepsilon_p(Q)=(-1, d(Q))_p$.

$\langle 3\rangle$ $r=15$, *and for every prime number p,* $-d(Q)\notin \boldsymbol{Q}_p^{*2}$ *or* $\varepsilon_p(Q)=(-1,-1)_p$.

$\langle 4\rangle$ $r\leq 14$.

(B) G coincides with a Dynkin graph which is obtained from one of the following 9 basic Dynkin graphs by elementary transformations repeated twice such that it has no vertices corresponding to short roots.

$$B_{17},\ D_{16},\ D_{12}+B_5,\ A_{15}+B_2,\ A_{11}+E_6,\ 2D_8,\ 2E_8,\ E_8+B_9,\ 2E_7+B_3.$$

Remarks. 1. $r=\operatorname{rank} Q=$ the number of vertices in G.

2. The symbol $\varepsilon_p(Q)\in\{+1,-1\}$ denotes the Hasse symbol of the inner product space $Q\otimes\boldsymbol{Q}$ over $\boldsymbol{Q}$. The symbol $(\ ,\ )_p$ is the Hilbert symbol. $\boldsymbol{Q}_p$

is the field of p-adic numbers and $Q_p^{*2}=\{a^2 \mid a \in Q_p,\ a\neq 0\}$. (cf. Serre [4])

3. If $G=G'+G''$ for some Dynkin graphs G' and G'', then $d(Q(G)) = d(Q(G'))d(Q(G''))$ and $\varepsilon_p(Q(G))=\varepsilon_p(Q(G'))\varepsilon_p(Q(G''))(d(Q(G')),\ d(Q(G'')))_p$.

4. $d(Q(A_k))=k+1$, $d(Q(D_l))=4$, $d(Q(E_6))=3$, $d(Q(E_7))=2$ and $d(Q(E_8)) =1$.

5. $\varepsilon_p(Q(A_k))=(-1,k+1)_p$ and $\varepsilon_p(Q(D_l))=\varepsilon_p(Q(E_m))=1$.

6. Assume that $a,\ b \in Q$ can be written in the form $a=p^\alpha u$, $b=p^\beta v$ with p-adic units $u,\ v \in Q_p$.

In case $p\neq 2$, $(a, b)_p=(-1)^{\alpha\beta\chi(p)}(u/p)^\beta(v/p)^\alpha$. Here $(\ /p)$ is the Legendre's quadratic residue symbol.

In case $p=2$, $(a, b)_p=(-1)^{\chi(u)\chi(v)+\alpha\omega(v)+\beta\omega(u)}$.

Here $\chi(n)\equiv(n-1)/2$, $\omega(n)\equiv(n^2-1)/8 \pmod 2$.

7. There are 9 kinds of positive definite unimodular lattices of rank 17. The root systems made of elements η with $\eta^2=2$ or 1 in them are of type B_{17}, $D_{16}+B_1$, $D_{12}+B_5$, $A_{15}+B_2$, $A_{11}+E_6$, $2D_8+B_1$, $2E_8+B_1$, E_8+B_9, $2E_7 +B_3$ respectively. This characterizes the basic 9 graphs. (An unnecessary component B_1 is omitted in the above.)

8. If there is a quartic surface in P^3 whose combination of singularities is G, then $r\leqq 19$.

In order to deal with cases to which we cannot apply Theorem 2, we would like to propose another notion.

Definition 3. Assume that applying the following procedure to the Dynkin graph G, we have obtained the Dynkin graph G'. Then we call the following procedure a *connection* of Dynkin graphs.

(1) Attach an integer to each vertex of G by the following rule: Now let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be the fundamental system of roots associated with a connected component G_1 of G. Let $\sum_{i=1}^k n_i\alpha_i$ be the associated maximal root. Then the attached integer to the vertex corresponding to α_i is n_i.

(2) Add one vertex and some edges to each component of G and make it the extended Dynkin graph of the corresponding type. Attach moreover the integer 1 to each new vertex.

(3) Choose in an arbitrary manner subsets A, B of the set of vertices of the extended graph satisfying the following conditions.

⟨a⟩ $A\cap B=\phi$.

⟨b⟩ The number of elements in B is at most 3.

⟨c⟩ Choose arbitrarily a component G_1 of G and let V be the set of vertices in G_1. Let N be the sum of attached numbers to elements in $B\cap V$. (If $B\cap V=\phi$, $N=0$.) Then, the greatest common divisor of N and the numbers attached to elements in $A\cap V$ is necessarily 1.

(4) Erase out all attached integers.

(5) Remove vertices belonging to A together with the edges issuing from them.

(6) Draw one new vertex. (We call this new vertex θ.) Connect θ

and each element in B following the next rule: If $v \in B$ corresponds to a long root, then we connect θ and v with a single edge $\overset{\theta}{\circ}\!-\!\!-\!\!-\!\overset{v}{\circ}$. If $v \in B$ corresponds to a short root, then we connect θ and v with a double edge with an arrow like this $\overset{\theta}{\circ}\!\Longrightarrow\!\overset{v}{\circ}$. (The arrow directs to v.)

Theorem 4. *Assume that G' is the Dynkin graph obtained by an elementary transformation* or *a connection from one of the basic* 9 *Dynkin graphs in Theorem* 2. *Assume moreover that applying an elementary transformation* or *a connection to G' once more, we have obtained a Dynkin graph G without vertices corresponding to short roots. Then there exists a quartic surface in the projective space of dimension* 3 *whose combination of singularities agrees with G.*

Note that by the Nikulin's lattice embedding theorem (Nikulin [3]) and by surjectivity of period mappings for $K3$ surfaces we can give a necessary and sufficient condition in order that there is a quartic surface with the combination of singularities G. But his theory does not give a systematic law. Moreover for concrete examples there is a very tiresome step if we apply his theory. (For complicated cases, it is tiresome to seek a suitable overlattice of $\boldsymbol{Z}\lambda \oplus Q(G)$ ($\lambda^2=-4$) which has a primitive embedding into an even unimodular lattice with signature (19, 3).)

With help of Nikulin's theory we can show the next corollary by our theorems.

Corollary 5. *Let $G=\sum a_k A_k+\sum b_l D_l+\sum c_m E_m$ be a Dynkin graph without any component of type B. Assume that the number of vertices $r=\sum a_k k+\sum b_l l+\sum c_m m$ is less than or equal to* 15. *Then there is a quartic surface in $\boldsymbol{P}^3$ with singularities G if and only if $G \neq A_6+9A_1$, $A_4+A_2+9A_1$, A_4+11A_1, $3A_2+9A_1$, $2A_2+11A_1$, A_2+13A_1, and $9A_1+E_6$.*

Conjecture. The converse of Theorem 4 is true.

Remarks. Let Λ denote an even unimodular lattice with signature (19, 3) or (18, 2). Let $\lambda \in \Lambda$ be an element with $\lambda^2=-4$. Let $M=\Lambda/\boldsymbol{Z}\lambda$. M has a natural bilinear form with values in $\boldsymbol{Q}$. If for any root sublattice $Q \subset M$ satisfying the following condition (*), there are a fundamental system of roots $\Delta \subset Q$ and elements $\alpha \in \Delta$, $u \in M$ such that $u \neq 0$, $u^2=0$, $u \cdot \alpha=1$ or 0, and $u \cdot \beta=0$ for every $\beta \in \Delta$ with $\beta \neq \alpha$, then the above conjecture is true.

(*) If an element $\eta \in M$ satisfies $\eta^2=2$ or 1 and if there is a non-zero integer m with $m\eta \in Q$, then η belongs to Q.

Indeed under the assumption $r=\sum a_k k+\sum b_l l+\sum c_m m \leqq 15$ we can check case by case that the conjecture is true. (We can make the list of Dynkin graphs G with $r=15$ such that $-d(Q(G)) \in \boldsymbol{Q}_p^{*2}$ and $\varepsilon_p(Q(G)) \neq (-1,-1)_p$ for some prime number p. It contains 44 items. We can check that every item in it except $2A_2+11A_1$ can be obtained by applying one connection after one elementary transformation from one of basic 9 Dynkin graphs.)

References

[1] Bourbaki, N.: Groupes et algèbre de Lie. chaps. 4–6, Hermann, Paris (1968).
[2] Durfee, A.: Fifteen characterizations of rational double points and simple critical points. Engein. Math. II, **25**, 131–163 (1979).
[3] Nikulin, V. V.: Integral symmetric bilinear forms and some of their applications. Math. USSR Izvestija, **14**, 103–167 (1980).
[4] Serre, J. P.: Còurs d'arithmétique. Presses Univ. France, Paris (1970).