73. On Sufficient Conditions for Convergence of Formal Solutions

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 14, 1985)

§1. **Introduction.** Let $x=(x_1, x_2)\in C^2$. For a multi-index $\alpha=(\alpha_1, \alpha_2)\in N^2$, $N=\{0, 1, 2, \cdots\}$, we set $(x\cdot\partial)^\alpha=(x_1\cdot\partial_1)^{\alpha_1}(x_2\cdot\partial_2)^{\alpha_2}$ where $\partial=(\partial_1\cdot\partial_2)$, $\partial_j=\partial/\partial x_j$, $j=1, 2$. Let $m\geqq 0$, $N\geqq 1$, $s\geqq 0$ be integers such that $0\leqq s\leqq m$ and let $s_1, \cdots, s_N$ be a set of integers such that $1=s_1\leqq s_2\leqq\cdots\leqq s_N$. In this note we are concerned with the convergence of all formal solutions of the equation

$$(P_0(x\cdot\partial)+Q_s(x\,;x\cdot\partial))u=f \tag{1.1}$$

where u denotes ${}^t(u_1, \cdots, u_N)$, $f={}^t(f_1, \cdots, f_N)$ is a given analytic vector function and the operators P_0 and Q_s are given by

$$P_0(x\cdot\partial)=\Big(\sum_{|\alpha|=m+s_j-s_k} a_\alpha^{jk}(x\cdot\partial)^\alpha\Big)_{\substack{j\downarrow 1,\dots,N\\ k\to 1,\dots,N}} \tag{1.2}$$

$$Q_s(x\,;x\cdot\partial)=\Big(\sum_{|\beta|\leqq m-s+s_j-s_k} b_\beta^{jk}(x)(x\cdot\partial)^\beta\Big)_{\substack{j\downarrow 1,\dots,N\\ k\to 1,\dots,N}}. \tag{1.3}$$

Here $a_\alpha^{jk}\in C$ and $b_\beta^{jk}(x)$ are analytic at $x=0$. If $s=0$, then we may assume that $b_\beta^{jk}(0)=0$ ($|\beta|=m+s_j-s_k$) in (1.1). Hence we assume this from now on.

Concerning this problem Kashiwara-Kawai-Sjöstrand showed the convergence of all formal solutions for a wider class of equations than (1.1) under the so-called ellipticity condition (cf. [2]). Here we show a new phenomenon when the ellipticity condition is not satisfied for equations belonging to a subclass of equations studied in [2]. Namely we shall introduce a new diophantine function $F_\sigma(t)$ and give a sufficient condition for the convergence of all formal solutions in terms of $F_\sigma(t)$. We note that this result is applied to the problem of holomorphic prolongation of solutions across characteristic points.

Finally the author would like to give thanks to the referee who gave the author useful suggestions in preparing this note.

§2. **Notations and results.** For $R>0$, $d\geqq 0$ let us define the set $\Gamma_{R,d}$ of holomorphic functions by

$$\Gamma_{R,d}=\{h(x)=\sum_{\gamma\geqq 0} h_\gamma x^\gamma/\gamma\,!\,;\ K>0 \text{ independent of } \gamma \text{ such that } |h_\gamma|\leqq K|\gamma|!\,R^{-|\gamma|}(|\gamma|+1)^{-d}\} \tag{2.1}$$

where $|h_\gamma|$ denotes the usual maximal norm of N-dimensional vector h_γ. For $\sigma\geqq 0$ we define the function $F_\sigma(t)$ of $t\in C$ by

$$F_\sigma(t)=\{\text{the set of all the cluster values of the sequence } \{\mu^\sigma(\nu/\mu-\tau)\} \text{ when } \nu,\ \mu\in N \text{ and } \nu,\ \mu\to\infty\}. \tag{2.2}$$

Remark. Obviously the function $F_\sigma(t)$ is multivalued in general. Here we list up some of its fundamental properties without proofs. The set $F_\sigma(t)$ is closed; $F_\sigma(t)=\phi$ if $t\notin[0,\infty)$, $F_\sigma(0)=[0,\infty)$ if $0<\sigma<1$, $=\phi$ if $\sigma\geqq 1$. In case $t>0$ is rational, $t=b/a$, $(a,b)=1$ then it follows that $F_\sigma(t)=\boldsymbol{R}^1$ $(0<\sigma<1)$, $=\{k/a\,;\,k\in\boldsymbol{Z}\}$ $(\sigma=1)$, $=\{0\}$ $(\sigma>1)$. On the other hand if $t>0$ irrational then $F_\sigma(t)=\boldsymbol{R}^1$ $(0<\sigma\leqq 1)$, $F_\sigma(t)\ni 0$ $(1<\sigma<2)$. In order to study the case $\sigma\geqq 2$ we expand t into the continued fraction $t=[a_0,a_1,a_2,\cdots]$ where $a_0=[t]$, $\alpha_0=t-a_0$, $a_n=[\alpha_n]$, $1/\alpha_{n+1}=\alpha_n-a_n$; $n=0,1,2,\cdots$. We define q_n by $q_{n+2}=a_nq_{n+1}+q_n$, $q_1=0$, $q_2=1$, $n=1,2,\cdots$. Then in case $\sigma>2$ the set $F_\sigma(t)$ is equal to the set of all the cluster values of the sequence $\{(-1)^{n-1}\cdot q_n^{\sigma-2}/a_{n-1}\}$ $(n=1,2,\cdots)$ when n tends to infinity.

Let $p_m(\eta)$ and $q_{m-s}(\eta)$ be the characteristic matrices of $P_0(x\cdot\partial)$ and $Q_s(0\,;x\cdot\partial)$ respectively i.e. the matrices which are obtained from (1.2) and (1.3) by replacing $x\cdot\partial$ with η and then setting $x=0$ and $|\beta|=m-s+s_j-s_k$. Let $\tau_p(p=1,\cdots,p_0)$ be the roots of the equation $\det p_m((t,1))=0$ and let m_p be its multiplicity.

We assume the following two conditions.

(A.1) $\qquad \det p_m(\eta)\neq 0 \quad \text{for} \quad \eta=(1,0) \quad \text{and} \quad (0,1).$

For a positive integer ν we define the set $F_\sigma(t)^\nu$ by $F_\sigma(t)^\nu=\{\tau^\nu\,;\,\tau\in F_\sigma(t)\}$. We take a circle C_p $(p=1,\cdots,p_0)$ in the complex plane $\boldsymbol{C}$ which encircles τ_p but no other $\tau_\mu(\mu\neq p)$. Then

(A.2)
$$\frac{1}{2\pi i}\int_{C_p}\operatorname{tr}\{(t-\tau_p)^{m_p-1}p_m((t,1))^{-1}q_{m-s}((t,1))\}dt\notin -F_{s/m_p}(\tau_p)^{m_p}$$

for $p=1,\cdots,p_0$ where tr in the integrand denotes the usual trace of a matrix and the integral is taken in the positive direction. Then

Theorem 2.1. *Suppose* (A.1) *and* (A.2). *Then there exists* $R_0>0$ *such that for any* $R\leqq R_0$, $d\geqq\max(3, s_N+1)$ *and for any* $f\in\Gamma_{R,d}$, *all formal solutions of* Eq. (1.1) *converge and are contained in* $\Gamma_{R,d-1-s_N}$.

Corollary 2.2. *Suppose* (A.1) *and* (A.2). *Then there exists* $R_0>0$ *such that for any* $0<R<R_0$ *the following holds: If* u *is holomorphic in a neighborhood of the origin and if* $(P_0+Q_s)u$ *is holomorphic in a neighborhood of* $|x_1|+|x_2|\leqq R$ *then* u *is holomorphic in a neighborhood of* $|x_1|+|x_2|\leqq R$.

Remark. Let $s\geqq 1$ be an integer and suppose that $\det p_m((\tau,1))=0$ for some $\tau\geqq 0$. Then it is easy to see that the surface $\phi(x)\equiv|x_1|+|x_2|=R$ $(R>0)$ is characteristic with respect to P_0+Q_s at the point x such that $|x_1|=\tau|x_2|$. The above theorem can be applied to this case.

Remark. If $Q_s\equiv 0$ then we can give a necessary and sufficient condition. We introduce, for $t\in\boldsymbol{C}$,

$$\rho(t)=\liminf_{\mu\to\infty}\,(\inf_{\nu\in N}|\mu t-\nu|^{1/\mu}).$$

The fundamental property of this function is studied in [1]. Then we can show that all formal solutions of the equation $P_0u=f$ converge if and only if $\rho(\tau)>0$ for all τ such that $\det p_m((\tau,1))=0$ and the condition (A.1) is satisfied.

References

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