# 72. Asymptotic Expansions of Solutions of Fuchsian Hyperbolic Equations in Spaces of Functions of Gevrey Classes

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In this paper, we deal with Fuchsian hyperbolic equations with Gevrey coefficients and establish the asymptotic expansions of solutions in spaces of functions of Gevrey classes. As to the Cauchy problem in Gevrey classes, see Tahara [3].

**1. Fuchsian hyperbolic equations.** Let us consider the following equation:

$$(\mathrm{E}) \qquad (t\partial_t)^m u + \sum_{\substack{j+|\alpha|\leqq m \\ j<m}} t^{l(j,\alpha)} a_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha u = 0,$$

where $(t, x)=(t, x_1, \cdots, x_n) \in [0, T]\times \boldsymbol{R}^n$ $(T>0)$, $m \in \boldsymbol{N}(=\{1, 2, \cdots\})$, $\alpha=(\alpha_1, \cdots, \alpha_n) \in \boldsymbol{Z}_+^n(=\{0, 1, 2, \cdots\}^n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $l(j, \alpha) \in \boldsymbol{Z}_+$ $(j+|\alpha|\leqq m$ and $j<m)$, $a_{j,\alpha}(t, x) \in C^\infty([0, T]\times \boldsymbol{R}^n)$ $(j+|\alpha|\leqq m$ and $j<m)$, $\partial_t=\partial/\partial t$, and $\partial_x^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$. Assume the following conditions:

(A-1) $l(j, \alpha) \in \boldsymbol{Z}_+$ $(j+|\alpha|\leqq m$ and $j<m)$ satisfy

$$\begin{cases} l(j,\alpha)=\kappa_1\alpha_1+\cdots+\kappa_n\alpha_n, & \text{when } j+|\alpha|=m \text{ and } j<m, \\ l(j,\alpha)>0, & \text{when } j+|\alpha|<m \text{ and } |\alpha|>0, \\ l(j,\alpha)\geqq 0, & \text{when } j+|\alpha|<m \text{ and } |\alpha|=0 \end{cases}$$

for some $\kappa_1, \cdots, \kappa_n \in \boldsymbol{Q}$ such that $\kappa_i>0$ $(i=1, \cdots, n)$.

(A-2) All the roots $\lambda_i(t, x, \xi)$ $(i=1, \cdots, m)$ of

$$\lambda^m + \sum_{\substack{j+|\alpha|=m \\ j<m}} a_{j,\alpha}(t,x)\lambda^j \xi^\alpha = 0$$

are *real*, *simple* and *bounded* on $\{(t, x, \xi) \in [0, T]\times \boldsymbol{R}^n \times \boldsymbol{R}^n; |\xi|=1\}$.

Then, (E) is one of the most fundamental examples of Fuchsian hyperbolic equations. The characteristic exponents $\rho=\rho_1(x), \cdots, \rho_m(x)$ are defined by the roots of

$$\rho^m + a_{m-1}(x)\rho^{m-1}+\cdots+a_0(x)=0,$$

where $a_j(x)=[t^{l(j,(0,\cdots,0))}a_{j,(0,\cdots,0)}(t,x)]|_{t=0}$ $(j=0, \cdots, m-1)$.

**2. Asymptotic expansions in $C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$.** Let $\mathcal{E}(\boldsymbol{R}^n)$ be the Schwartz space on $\boldsymbol{R}^n$ and let $C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$ be the space of all $C^\infty$ functions on $(0, T)$ with values in $\mathcal{E}(\boldsymbol{R}^n)$. Then, by applying the result in Tahara [2] we have

**Theorem 1.** *Assume that* (A-1), (A-2) *and the condition*:

(T) $\quad l(j, \alpha)\geqq \kappa_1\alpha_1+\cdots+\kappa_n\alpha_n,$ *when* $j+|\alpha|<m$ *and* $|\alpha|>0$

*hold, and that* $\rho_i(x)-\rho_j(x) \notin \boldsymbol{Z}$ *holds for any* $x \in \boldsymbol{R}^n$ *and* $1\leqq i\neq j\leqq m$. *Then, we have the following results.*

(1) *Any solution* $u(t, x) \in C^\infty((0, T), \mathcal{E}(\boldsymbol{R}^n))$ *of* (E) *can be expanded*

*asymptotically into the form*

$$(*)\qquad u(t,x)\sim\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big)$$

*as* $t\to+0$ *for some unique* $\varphi_i(x),\ \varphi_{k,h}^{(i)}(x)\in\mathcal{E}(\boldsymbol{R}^n)$.

(2) *Conversely, for any* $\varphi_1(x),\cdots,\varphi_m(x)\in\mathcal{E}(\boldsymbol{R}^n)$ *there exist a unique solution* $u(t,x)\in C^\infty((0,T),\mathcal{E}(\boldsymbol{R}^n))$ *of* (E) *and unique coefficients* $\varphi_{k,h}^{(i)}(x)\in\mathcal{E}(\boldsymbol{R}^n)$ $(1\leqq i\leqq m,\ 1\leqq k<\infty$ *and* $0\leqq h\leqq mk)$ *such that the asymptotic relation in* (1) *holds.*

In Theorem 1, the condition (T) seems to be essential to the characterization of the general solution in $C^\infty((0,T),\mathcal{E}(\boldsymbol{R}^n))$. Therefore, if we want to consider the case without (T), we must treat the equation (E) in suitable subclasses of $C^\infty((0,T),\mathcal{E}(\boldsymbol{R}^n))$.

**3. Asymptotic expansions in $C^\infty((0,T),\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$.** A function $f(x)$ $(\in C^\infty(\boldsymbol{R}^n))$ is said to belong to the *Gevrey class* $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$, if $f(x)$ satisfies the following; for any compact subset $K$ of $\boldsymbol{R}^n$ there are $C>0$ and $h>0$ such that

$$(3.1)\qquad \sup_{x\in K}|\partial_x^\alpha f(x)|\leqq C\,h^{|\alpha|}(|\alpha|!)^s\qquad\text{for any }\alpha\in\boldsymbol{Z}_+^n.$$

We denote by $C^\infty((0,T),\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ [resp. $C^\infty([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$] the space of all $C^\infty$ functions on $(0,T)$ [resp. $[0,T]$] with values in $\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ equipped with the locally convex topology in Komatsu [1].

Now, let us consider the equation (E) in $C^\infty((0,T),\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ under (A-1) and (A-2). Let $l(j,\alpha)$ $(j+|\alpha|<m$ and $|\alpha|>0)$ and $\kappa_1,\cdots,\kappa_n$ be as in (A-1). Define the irregularity index $\sigma$ $(\geqq1)$ by

$$\sigma=\max\Big[1,\ \max_{\substack{j+|\alpha|<m\\|\alpha|>0}}\Big\{\min_{\tau\in\mathfrak{S}_n}\Big(\max_{1\leqq r\leqq n}M_{j,\alpha}(\tau,r)\Big)\Big\}\Big],$$

where $\mathfrak{S}_n$ is the permutation group of $n$-numbers and

$$M_{j,\alpha}(\tau,r)=\frac{\sum_{i=1}^{r}(\kappa_{\tau(i)}-\kappa_{\tau(r)})\alpha_{\tau(i)}+(m-j)\kappa_{\tau(r)}-l(j,\alpha)}{(m-j-|\alpha|)\kappa_{\tau(r)}}.$$

Impose the following conditions:

(A-3) $1<s<\sigma/(\sigma-1)$.

(A-4) $a_{j,\alpha}(t,x)\in C^\infty([0,T],\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ $(j+|\alpha|\leqq m$ and $j<m)$.

When $\sigma=1$, (A-3) is read $1<s<\infty$. Then, we have

**Theorem 2.** *Assume that* (A-1)–(A-4) *hold and that* $\rho_i(x)-\rho_j(x)\notin\boldsymbol{Z}$ *holds for any* $x\in\boldsymbol{R}^n$ *and* $1\leqq i\neq j\leqq m$. *Then, we have the following results.*

(1) *Any solution* $u(t,x)\in C^\infty((0,T),\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *of* (E) *can be expanded asymptotically into the form*

$$(**)\qquad u(t,x)\sim\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{\infty}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big)$$

*as* $t\to+0$ *for some unique* $\varphi_i(x),\ \varphi_{k,h}^{(i)}(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$.

(2) *Conversely, for any* $\varphi_1(x),\cdots,\varphi_m(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ *there exist a unique solution* $u(t,x)\in C^\infty((0,T),\mathcal{E}^{\{s\}}(\boldsymbol{R}^n))$ *of* (E) *and unique coefficients* $\varphi_{k,h}^{(i)}(x)\in\mathcal{E}^{\{s\}}(\boldsymbol{R}^n)$ $(1\leqq i\leqq m,\ 1\leqq k<\infty$ *and* $0\leqq h\leqq mk)$ *such that the asymptotic relation in* (1) *holds.*

Here, the meaning of the asymptotic relation (**) [resp. (*)] is as follows; for any $a>0$ and any compact subset $K$ of $\boldsymbol{R}^n$ there is an $N_0 \in \boldsymbol{N}$ such that for any $N \geqq N_0$

$$t^{-a}(t\partial_t)^l\Big[u(t,x)-\sum_{i=1}^{m}\Big(\varphi_i(x)t^{\rho_i(x)}+\sum_{k=1}^{N}\sum_{h=0}^{mk}\varphi_{k,h}^{(i)}(x)t^{\rho_i(x)+k}(\log t)^{mk-h}\Big)\Big]\Big|_K$$
$$\to 0 \text{ in } \mathcal{E}^{\{s\}}(K) \text{ [resp. } \mathcal{E}(K)] \text{ as } t\to+0 \text{ for any } l\in \boldsymbol{Z}_+,$$

where $\mathcal{E}^{\{s\}}(K)$ is the locally convex space of all functions $f(x) \in C^\infty(K)$ satisfying (3.1) for some $C>0$ and $h>0$ (see Komatsu [1]).

**Remark.** (1) $\sigma=1$ is equivalent to (T).

(2) When $\kappa_1=\cdots=\kappa_n$ $(=\kappa_*)$, $\sigma$ is given by

$$\sigma=\max\Big[1,\ \max_{\substack{j+|\alpha|<m\\ |\alpha|>0}}\Big(\frac{m-j-l(j,\alpha)/\kappa_*}{m-j-|\alpha|}\Big)\Big].$$

(3) Also in the case $l(j,\alpha)\in \boldsymbol{Q}$ $(j+|\alpha|\leqq m$ and $j<m)$, we can obtain the same results as above. To see this, we have only to apply the change of variables $t^{1/N}\to t$ and $x\to x$. See § 7 of Tahara [2].

(4) The reason why the logarithmic terms appear in (*) or (**) (notwithstanding the assumption $\rho_i(x)-\rho_j(x)\notin \boldsymbol{Z}$ for $i\neq j$) lies in the following formula: $(\partial/\partial x_i)t^{\rho(x)}=(\partial\rho(x)/\partial x_i)t^{\rho(x)}(\log t)$. Therefore, if $\rho_i(x)$ $(1\leqq i\leqq m)$ are constant on $U$, the logarithmic terms do not appear on $U$ in (*) or (**).

**4. Examples.** (1) Let $P_1$ be of the form

$$P_1=(t\partial_t)^2-t^{2\kappa}\partial_x^2+t^l a(t,x)\partial_x+b(t,x)(t\partial_t)+c(t,x),$$

where $(t,x)\in[0,T]\times\boldsymbol{R}$ and $2\kappa,\ l\in\boldsymbol{N}$. Then, $\sigma$ is given by

$$\sigma=\max\Big\{1,\ \frac{2\kappa-l}{\kappa}\Big\}.$$

(2) Let $P_2$ be of the form

$$P_2=(t\partial_t)^2-t^{2\kappa_1}\partial_{x_1}^2-t^{2\kappa_2}\partial_{x_2}^2+t^{l_1}a_1(t,x)\partial_{x_1}+t^{l_2}a_2(t,x)\partial_{x_2}+b(t,x)(t\partial_t)+c(t,x),$$

where $(t,x)\in[0,T]\times\boldsymbol{R}^2$ and $2\kappa_1,\ 2\kappa_2,\ l_1,\ l_2\in\boldsymbol{N}$. Then, $\sigma$ is given by

$$\sigma=\max\Big\{1,\ \frac{2\kappa_1-l_1}{\kappa_1},\ \frac{2\kappa_2-l_2}{\kappa_2}\Big\}.$$

(3) Let $P_3$ be of the form

$$P_3=(t\partial_t)((t\partial_t)^2-t^{2\kappa_1}\partial_{x_1}^2-t^{2\kappa_2}\partial_{x_2}^2)+t^l a(t,x)\partial_{x_1}\partial_{x_2},$$

where $(t,x)\in[0,T]\times\boldsymbol{R}^2$ and $2\kappa_1,\ 2\kappa_2,\ l\in\boldsymbol{N}$. Then, $\sigma$ is given by

$$\sigma=\begin{cases}\max\Big\{1,\ \dfrac{3\kappa_1-l}{\kappa_1},\ \dfrac{\kappa_1+2\kappa_2-l}{\kappa_2}\Big\}, & \text{when } 0<\kappa_1\leqq\kappa_2,\\[2ex] \max\Big\{1,\ \dfrac{2\kappa_1+\kappa_2-l}{\kappa_1},\ \dfrac{3\kappa_2-l}{\kappa_2}\Big\}, & \text{when } 0<\kappa_2\leqq\kappa_1.\end{cases}$$

Details and proofs will be published elsewhere.

## References

[1] Komatsu, H.: Ultradistributions, I. Structure theorems and a characterization. J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **20**, 25–105 (1973).

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[3] ——: Cauchy problems for Fuchsian hyperbolic equations in spaces of functions of Gevrey classes. Proc. Japan Acad., **61A**, 63–65 (1985).