70. On Riemann Type Integral of Functions with Values in a Certain Fréchet Space

By Yumiko SATO
Department of Mathematics, Mie University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 14, 1985)

1. Introduction. Lex X be a Fréchet space [1] [5] with quasi-norm $\| \ \|$ such that, for every $x \in X$ and real number a, $\|ax\|=|a|^{\alpha}\|x\|$ holds for some fixed α, $0<\alpha<1$. We want to consider some sort of integrals of functions defined on a bounded closed interval and taking values in this space. But the theory of the Bochner integral does not apply, since X is not a Banach space, nor is the theory of Riemann integrals extended to this case because of slowness of the convergence $\|ax\|\to 0$ as $a\to 0$. In this paper we prove that Riemann type integrals exist for Hölder continuous functions with exponent γ if $\gamma>1-\alpha$, and we give an upper bound of the norm of the integral in terms of γ and Hölder constant. This integral is motivated by the problem of canonical representations of stationary symmetric α-stable processes.

2. Theorems. Let X be a Fréchet space with the property stated above and x_t be a function of $t\in I=[a, b]$ which has values in X. Sometimes we write $x_t=x(t)$.

Definition 1. Let γ, δ_0, K be positive numbers. We call x_t satisfies Condition $C_\gamma(\delta_0, K)$ if $\|x_t-x_s\|\leqq K|t-s|^{\gamma}$ whenever $t, s\in I$ and $|t-s|\leqq\delta_0$.

Let $\{I_i, 1\leqq i\leqq n\}$ be a partition of I such that $a=a_0<a_1<\cdots<a_n=b$, $I_i=[a_{i-1}, a_i]$. A pair of $\{I_i\}$ and $\{t_i\}$, $t_i\in I_i$, is denoted by $S=(\{I_i\}, \{t_i\})$. The length of I_i is denoted by $|I_i|$.

Definition 2. Suppose that x_t is a function defined on I. We say that x_t is Riemann type integrable over I if there is an element $\mathcal{J}$ in X with the following property: For each $\varepsilon>0$, there is $\delta>0$ such that

$$\left\|\sum_{i=1}^{n}|I_i|x(t_i)-\mathcal{J}\right\|<\varepsilon$$

whenever $S=(\{I_i\}, \{t_i\})$ satisfies $\max_{1\leq i\leq n}|I_i|<\delta$. We call $\mathcal{J}$ Riemann type integral and write $\mathcal{J}=\int_I x_t dt$.

Then we have the following theorems.

Theorem 1. *If x_t satisfies Condition $C_\gamma(\delta_0, K)$ for some δ_0, K and γ such that $1\geqq\gamma>1-\alpha$, then x_t is Riemann type integrable over I.*

Theorem 2. *Under the same conditions as Theorem 1, we have the following inequality:*

$$\left\|\int_I x_t dt\right\|\leqq M^{1-\alpha}|I|^{\alpha}\sup_{t\in I}\|x_t\|+M^{-\rho}|I|^{\alpha+\gamma}KA_{\alpha\gamma}$$

where $\rho=\alpha+\gamma-1$, $A_{\alpha\gamma}=2^{1-2\alpha}2^{\rho}/(2^{\rho}-1)+2^{\gamma}$ and M is any number bigger than $2|I|/\delta_0$.

3. Proof of Theorems. Given $S=(\{I_i\},\{t_i\})$, let $\mathcal{J}_S=\sum_{i=1}^n |I_i| x(t_i)$. We have to evaluate $\|\mathcal{J}_S-\mathcal{J}_{S'}\|$ for distinct S and S'. In order to do this we begin with modification of $\mathcal{J}_S$ for a fixed S.

Step 1. Fix $S=(\{I_i, 1\leq i\leq n\},\{t_i\})$, $I_i=[a_{i-1}, a_i]$. We make from S three auxiliary partitions $\{J_k^p\}$, $\{I_j^p\}$ and $\{F_{k1}^p, F_{k2}^p\}$ as follows.

i) First, fix an integer M such that $|I|/M\geq \max_{1\leq i\leq n}|I_i|$. For each nonnegative integer p, $\{J_k^p\}$ is a partition of I into 2^pM subintervals of equal length. Namely,

$$J_k^p=[c_{k-1}, c_k],\quad k=1,\cdots,2^pM.\quad c_k=a+k|I|/2^pM.$$

ii) Let $\{I_j^p, j=1,2,\cdots,p'\}$ be the superposition of $\{I_i\}$ and $\{J_k^p\}$. Numbering of $I_1^p, I_2^p,\cdots, I_{p'}^p$ is from left to right. We have $p'<n+2^pM$.

iii) Each interval J_k^p is the union of some intervals from $\{I_j^p\}$. Denote $J_k^p=I_{k'}^p\cup I_{k'+1}^p\cup\cdots\cup I_{k'+k''}^p$. In case $k''\geq 1$, divide each J_k^p into two subintervals F_{k1}^p and F_{k2}^p, where $F_{k1}^p=I_{k'}^p$ and $F_{k2}^p=\cup_{\tau=1}^{k''} I_{k'+\tau}^p$. In case $k''=0$, let $F_{k1}^p=J_k^p$, $F_{k2}^p=\phi$.

There is a finite number N such that every J_k^N consists of at most two intervals from $\{I_j^N\}$.

Let $\mathcal{J}_S^{0*}=\sum_{k=1}^M |J_k^0| x(s_k^0)$ and $\mathcal{J}_S^p=\sum_{k=1}^{2^pM}\{|F_{k1}^p|x(s_{k1}^p)+|F_{k2}^p|x(s_{k2}^p)\}$ where s_k^0, s_{k1}^p and s_{k2}^p are taken from the original $\{t_i\}$ as follows: i) Choose I_i that includes $I_{k'}^0$ and let $s_k^0=t_i$. ii) Choose I_i that includes F_{k1}^p and let $s_{k1}^p=t_i$. iii) Choose I_i that includes $I_{k'+1}^p$ and let $s_{k2}^p=t_i$. Notice that s_k^0, s_{k1}^p, s_{k2}^p are not always contained in J_k^0 F_{k1}^p, F_{k2}^p, respectively. It is easily seen that $\mathcal{J}_S=\mathcal{J}_S^N$.

Step 2. If $\max|I_i|$ is small enough, we can choose M that satisfies $\max|I_i|<|I|/M<\delta_0/2$. Then from Condition $C_\gamma(\delta_0, K)$ we get the following inequalities:

$$\begin{aligned}\|\mathcal{J}_S^{0*}-\mathcal{J}_S^0\|&=\left\|\sum_{k=1}^M |J_k^0|x(s_k^0)-\sum_{k=1}^M(|F_{k1}^0|x(s_{k1}^0)+|F_{k2}^0|x(s_{k2}^0))\right\|\\&=\left\|\sum_{k=1}^M |F_{k2}^0|(x(s_k^0)-x(s_{k2}^0))\right\|\leq M(|I|/M)^\alpha K(2|I|/M)^\gamma=2^\gamma K|I|^{\alpha+\gamma}M^{-\rho}.\end{aligned}$$

We have

$$J_k^p=F_{k1}^p\cup F_{k2}^p=J_l^{p+1}\cup J_{l+1}^{p+1}=F_{l1}^{p+1}\cup F_{l2}^{p+1}\cup F_{l+11}^{p+1}\cup F_{l+12}^{p+1},$$

where $l=2(k-1)+1$. Moreover, either $F_{k1}^p\supset J_l^{p+1}$ and $F_{l2}^{p+1}=\phi$ or $F_{k1}^p=F_{l1}^{p+1}$. Hence,

$$\begin{aligned}\|\mathcal{J}_S^p-\mathcal{J}_S^{p+1}\|&=\left\|\sum_{k=1}^{2^pM}\{|F_{l+11}^{p+1}|(x(s_{k2}^p)-x(s_{l+11}^{p+1}))+|F_{l+12}^{p+1}|(x(s_{k2}^p)-x(s_{l+12}^{p+1}))\}\right\|\\&\leq\sum_{k=1}^{2^pM}\{|F_{l+11}^{p+1}|^\alpha K|s_{k2}^p-s_{l+11}^{p+1}|^\gamma+|F_{l+12}^{p+1}|^\alpha K|s_{k2}^p-s_{l+12}^{p+1}|^\gamma\}\\&\leq\sum_{k=1}^{2^pM}K|J_k^p|^\gamma\{|F_{l+11}^{p+1}|^\alpha+|F_{l+12}^{p+1}|^\alpha\}\leq 2^pMK(|I|/2^pM)^\gamma 2(|I|/2^{p+2}M)^\alpha\\&=2^{-p\rho}2^{-2\alpha+1}K|I|^{\alpha+\gamma}M^{-\rho}.\end{aligned}$$

Here we used the fact if $a\geq 0$, $b\geq 0$ and $a+b=1$ then $a^\alpha+b^\alpha\leq 2(1/2)^\alpha$ for $0<\alpha\leq 1$. Now we have

$$\begin{aligned}\|\mathcal{J}_S^0-\mathcal{J}_S^N\|&\leq\|\mathcal{J}_S^0-\mathcal{J}_S^1\|+\|\mathcal{J}_S^1-\mathcal{J}_S^2\|+\cdots+\|\mathcal{J}_S^{N-1}-\mathcal{J}_S^N\|\\&\leq 2^{1-2\alpha}K|I|^{\alpha+\gamma}M^{-\rho}\{1+2^{-\rho}+2^{-2\rho}+\cdots+2^{-(N-1)\rho}\}\\&<2^{1-2\alpha}K|I|^{\alpha+\gamma}M^{-\rho}2^\rho/(2^\rho-1).\end{aligned}$$

Step 3. Let $S=(\{I_i\},\{t_i\})$ and $S'=(\{I'_j\},\{t'_j\})$. Assume that we can take an integer M such that both $\max|I_i|$ and $\max|I'_j|$ are less than $|I|/M$ and $|I|/M<\delta_0/2$. First we note that

$$\|\mathcal{I}_S^{0*}-\mathcal{I}_{S'}^{0*}\|=\left\|\sum_{k=1}^{M}|J_k^0|\{x(s_k^0)-x(s_{k'}^0)\}\right\|$$
$$\leq\sum_{1}^{M}(|I|/M)^{\alpha}K(2|I|/M)^{\gamma}=2^{\gamma}K|I|^{\alpha+\gamma}M^{-\rho}.$$

Using this and the inequalities of step 2, we have

$$\|\mathcal{I}_S-\mathcal{I}_{S'}\|\leq\|\mathcal{I}_S-\mathcal{I}_S^0\|+\|\mathcal{I}_S^0-\mathcal{I}_S^{0*}\|+\|\mathcal{I}_S^{0*}-\mathcal{I}_{S'}^{0*}\|$$
$$+\|\mathcal{I}_{S'}^{0*}-\mathcal{I}_{S'}^0\|+\|\mathcal{I}_{S'}^0-\mathcal{I}_{S'}\|$$
$$\leq K|I|^{\alpha+\gamma}M^{-\rho}\{4^{1-\alpha}2^{\rho}/(2^{\rho}-1)+3\cdot 2^{\gamma}\}.$$

It follows that for any $\varepsilon>0$, there is a $\delta>0$ such that $\|\mathcal{I}_S-\mathcal{I}_{S'}\|<\varepsilon$. Thus, by usual argument, Theorem 1 is proved.

Step 4. For any $S=(\{I_i\},\{t_i\})$ and M such that $\max|I_i|\leq|I|/M<\delta_0/2$, we have

$$\|\mathcal{I}_S\|\leq\|\mathcal{I}_S^{0*}\|+\|\mathcal{I}_S-\mathcal{I}_S^0\|+\|\mathcal{I}_S^0-\mathcal{I}_S^{0*}\|$$
$$\leq\left\|\sum_{k=1}^{M}(|I|/M)x(s_k^0)\right\|+K|I|^{\alpha+\gamma}M^{-\rho}\{2^{1-2\alpha}2^{\rho}/(2^{\rho}-1)+2^{\gamma}\}$$
$$\leq M^{1-\alpha}|I|^{\alpha}\sup_{t\in I}\|x_t\|+M^{-\rho}|I|^{\alpha+\gamma}KA_{\alpha\gamma}$$

where $A_{\alpha\gamma}=2^{1-2\alpha}2^{\rho}/(2^{\rho}-1)+2^{\gamma}$. This shows Theorem 2.

4. **Application.** Let $\{x_t, -\infty<t<\infty\}$ be a symmetric α-stable ($s\alpha s$) process, $0<\alpha<1$. That is, any finite linear combination $y=\sum_{i=1}^{n}c_i x_{t_i}$ has a characteristic function of the form $\varphi_y(u)=\exp(-a_y|u|^{\alpha})$, $a_y\geq 0$. We define $\|y\|=a_y$. It is known that this is a quasi-norm and convergence defined by it is equivalent to convergence in probability [3]. The space of all such linear combinations and their limits in probability is denoted by X. Any element $x\in X$ has $s\alpha s$ distribution and thus the quasi-norm is extended to X. This is a Fréchet space of our type.

When $\{x_t\}$ is stationary and admits a prediction [4], we can use our integral to construct a canonical stochastic measure under some supplementary conditions and extend Urbanik's results [2]. Note that $E|x_t|=\infty$ in our case, while Urbanik's theory uses Banach space arguments, assuming existence of finite expectations.

References

[1] N. Dunford and J. Scwartz: Linear Operators, Part I. Interscience (1958).
[2] Y. Sato: A linear prediction problem for symmetric α-stable processes with $1/2<\alpha<1$ (to appear in J. Math. Soc. Japan).
[3] M. Schilder: Some structure theorems for the symmetric stable laws. Ann. of Math. Stat., **41**, no. 2, 412–421 (1970).
[4] K. Urbanik: Some prediction problems for strictly stationary processes. Proc. of the Fifth Berkeley Symp. vol. II, part 1, Univ. of California Press, pp. 235–258 (1967).
[5] K. Yosida: Functional Analysis. Springer-Verlag (1965).