69. *First Hitting Time for Bessel Processes*

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By a Bessel process with index α ($\alpha>0$), we mean a conservative diffusion process on the half line $[0, \infty)$ determined by the generator

$$A=\frac{1}{2}\left(\frac{d^2}{dx^2}+\frac{\alpha-1}{x}\frac{d}{dx}\right).$$

In the case $0<\alpha<2$, an appropriate boundary condition must be imposed at the origin. In this note we restrict ourselves to the reflecting barrier case.

The following theorem for the d-dimensional Brownian motion is well known. Let σ_r denote the first exit time from the ball B_r with center 0 and radius r. Suppose $\|x\|\leqq r$, where $\|x\|$ denotes the Euclidean norm of x. Then the expect time spent in B_r by Brownian motion starting at x is given by

$$E_x(\sigma_r)=\frac{r^2-\|x\|^2}{d}.$$

The object of this note is to extend this result to the Bessel processes with reflecting barrier, replacing d by general α. Further we will derive explicitly the second moment of the first passage time to the point r.

Let T_r denote the first hitting time of the point r by the Bessel process $X(t)$, that is,

$$T_r=\inf\{t>0:\ X(t)=r\}.$$

Proposition 1. *Consider points $a<x<b$. Then we have*

$$P_x(T_a<T_b)=\begin{cases}\dfrac{x^{2-\alpha}-b^{2-\alpha}}{a^{2-\alpha}-b^{2-\alpha}} & \text{if } \alpha\neq 2\\[2ex] \dfrac{\log b-\log x}{\log b-\log a} & \text{if } \alpha=2.\end{cases}$$

Proof. Let $S(x)$ be a scale function for a regular diffusion on an interval I of the line. Therefore S is a strictly increasing function such that if $a<x<b$ and $a, b\in I^\circ$ (here I° is the interior of I), and the probability for the process reaching a before b is

$$P_x(T_a<T_b)=\frac{S(b)-S(x)}{S(b)-S(a)}.$$

We may take $S(x)=\log x$ if $\alpha=2$, $S(x)=(2-\alpha)^{-1}x^{2-\alpha}$ if $\alpha\neq 2$ and so the desired formula is obtained.

By the standard argument in Markov process (K. Ito [2], F. B. Knight [4]), we obtain

Corollary. (i) *When $\alpha>2$, the Bessel process is transient and*

$$P_x\{\lim_{t\to\infty} X(t)=\infty\}=1.$$

(ii) *When $0<\alpha\leqq 2$, the reflecting Bessel process is recurrent. If I is any interval, then for every N, $P_x\{X(t)\in I$ for some $t>N\}=1$.*

Theorem 1. *Suppose $\alpha>0$ and $x\leqq r$. Then the expect time for reaching at the point r is as follows*

$$E_x(T_r)=\frac{r^2-x^2}{\alpha}.$$

Proof. We adapt the proof by Karlin-Taylor [3] for the Brownian motion. Recall that the domain of the infinitesimal generator A includes at least those functions f having continuous second derivatives for which f and Af converge to zero as $x\to\infty$. Then

$$Af=\frac{1}{2}\left(\frac{d^2f}{dx^2}+\frac{\alpha-1}{x}\frac{df}{dx}\right). \tag{1}$$

Now set $u(x)=r^2-x^2$ for $x\leqq r$ and extended to $x>r$ to be twice continuously differentiable and to vanish at infinity. Apply (1) to u noting by continuity of paths that $X(T_r)=r$ and so $u(X(T_r))=0$. Further, for $t<T_r$, $u(X(t))=r^2-(X(t))^2$ and at these times $Au(X(t))=-\alpha$. We do not know a priori that $E_x(T_r)<\infty$, hence we cannot directly utilize Dynkin formula

$$E_x\left[\int_0^\sigma Au(X(t))dt\right]=E_x[u(X(\sigma))]-u(x), \tag{2}$$

where σ is a stopping time with finite expectation and u is in $\mathcal{D}(A)$.

We define a sequence of approximating stopping times. For each positive integer N, let $T_N=T_r\wedge N=\min(T_r, N)$. Obviously T_N is bounded and $E_x(T_N)\leqq N<\infty$. With Corollary in mind we see that every sample path escapes the interval $[0, r]$ with probability one regardless of transient or recurrence. Therefore T_N increases to T_r as $N\to\infty$. In virtue of the Dynkin formula to T_N and the function u defined above, we have

$$E_x[u(X(T_N))]-u(x)=E_x\left[\int_0^{T_N} Au(X(t))dt\right]=-\alpha E_x[T_N].$$

Therefore $E_x[T_N]\leqq(2/n)\|u\|$. By monotone convergence, T_N increases and we obtain that $E_x[T_r]\leqq(2/n)\|u\|$. Now we can apply (2) with T_r as σ and since $u(X(T_r))=0$, the Dynkin formula yields $u(x)=\alpha E_x[T_r]$ and then our Theorem 1 ensues.

To prove Theorem 2 we need the following extension of Dynkin formula.

Proposition 2. *Let f, Af, $A^2f\in\mathcal{D}(A)$ and $f(X(t))$, $Af(X(t))$, $A^2f(X(t))$ be right continuous in t. Let τ be a stopping time such that $E_x[\tau]<\infty$. Then*

$$E_x[f(X(\tau))]=f(x)+E_x[\tau Af(X(\tau))]-E_x\left[\int_0^\tau sA^2f(X(s))ds\right] \tag{3}$$

[K. B. Athreya-T. G. Kurtz [1]].

Theorem 2. *Suppose $\alpha>0$ and $x\leqq r$. Then the second moment of the expected time for hitting the point r is given by*

$$E_x(T_r^2)=\frac{4r^2(r^2-x^2)}{(2+\alpha)\alpha^2}+\frac{(r^2-x^2)^2}{\alpha(2+\alpha)}.$$

Proof. A fully rigorous proof requires truncating T_r by $T_N=T_r\wedge N$ and operating in terms of T_N as we did in the analysis of Theorem 1. We omit this technical point and proceed as if $E_x[T_r^2]<\infty$.

We apply Proposition 2 to the function

$$f(x)=\begin{cases}(r^2-x^2)^2 & \text{if } x\leqq r\\ \text{very smooth} & \text{if } x>r\end{cases}$$

with $f(x)$ vanishing rapidly as $x\to\infty$. Recall the formula (1) for the generator, then we have

$$Af(x)=-2\alpha(r^2-x^2)+4x^2 \qquad \text{for } x\leqq r$$

and

$$A^2f(x)=(2\alpha+4)\alpha \qquad \text{for } x\leqq r.$$

The generalization (3) of Dynkin formula gives

$$0=(r^2-x^2)^2+E_x[4r^2T_r]-E_x\left[(4+2\alpha)\alpha\frac{T_r^2}{2}\right]. \tag{4}$$

We substitute $\frac{r^2-x^2}{\alpha}$ for $E_x[T_r]$ in (4) to get the desired result.

References

[1] K. B. Athreya and T. G. Kurtz: A generalization of Dynkin's identity and some applications. Ann. of Prob., 1, 570–579 (1973).

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[3] S. Karlin and H. M. Taylor: A Second Course in Stochastic Processes. Academic Press, New York-London (1981).

[4] F. B. Knight: Essentials of Brownian Motion and Diffusion. American Mathematical Society, Providence (1981).