68. Upper Semicontinuity of Eigenvalues of Selfadjoint Operators Defined on Moving Domains

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1. Introduction. We are interested in "wild" perturbations in the sense of J. Rauch and M. Taylor [6], on eigenvalue problems for the Laplacian. We show the upper semicontinuity of each k-th eigenvalue of the minus Laplacian with respect to a domain perturbation belonging to a certain class. This class contains a perturbation argued by the author [5]. Hereafter we describe all statements only in an abstract fashion.

Let X and V_ε be real, separable and infinitely dimensional Hilbert spaces with $X \supset V_\varepsilon$. We assume that the injection $V_\varepsilon \to X$ is compact. We denote by $|\ |$ and $(\ ,\ )$ the norm and inner product on X, respectively. Here ε means the value zero or the values of a sequence decreasing to zero. Let $a_\varepsilon : V_\varepsilon \times V_\varepsilon \to \boldsymbol{R}$ be a symmetric continuous bilinear form such that $a_\varepsilon(v) \geqq c_\varepsilon \|v\|_V^2$ for all $v \in V_\varepsilon$, where $a_\varepsilon(v) = a_\varepsilon(v, v)$ and c_ε is a positive constant. We denote by H_ε the closure of V_ε in X and denote by P_ε the orthogonal projection from X onto H_ε. We set $\Sigma = \{x \in X \mid |x| = 1\}$. We define a positive selfadjoint operator $A_\varepsilon : D(A_\varepsilon) \to H_\varepsilon$ by $a_\varepsilon(u, v) = (A_\varepsilon u, v)$ for all $u \in D(A_\varepsilon)$ and $v \in V_\varepsilon$, where $D(A_\varepsilon) = \{u \in V_\varepsilon \mid {}^\exists c > 0$ such that $|a_\varepsilon(u,v)| \leqq c|v|$ for all $v \in V_\varepsilon\}$. We consider the equation: $A_\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon$, $\mu_\varepsilon \in \boldsymbol{R}$ and $u_\varepsilon \in \Sigma$. Let $\mu_\varepsilon^{(k)}$ be the k-th eigenvalue of A_ε counting with its multiplicity; $0 \leqq \mu_\varepsilon^{(1)} \leqq \mu_\varepsilon^{(2)} \leqq \cdots$ and $\mu_\varepsilon^{(k)} \to \infty$ as $k \to \infty$. We have

$$(1)\qquad \begin{aligned} \mu_\varepsilon^{(1)} &= \inf_{V_\varepsilon \cap \Sigma \ni x} a_\varepsilon(x) \\ \mu_\varepsilon^{(k)} &= \sup_{\substack{H_\varepsilon \ni x_i \\ 1 \leqq i \leqq k-1}} \ \inf_{\substack{V_\varepsilon \cap \Sigma \ni x \\ (x, x_i) = 0, 1 \leqq i \leqq k-1}} a_\varepsilon(x) \quad k \geqq 2 \end{aligned}$$

(cf. R. Courant and D. Hilbert [4]). If $\varepsilon = 0$ then we drop from V_ε, H_ε, A_ε, P_ε and so on. Next we describe our result.

Theorem 1. *If*

$$(2)\qquad \operatorname*{s-lim}_{\varepsilon \to 0} (1 + \lambda A_\varepsilon)^{-1} P_\varepsilon = (1 + \lambda A)^{-1} P$$

for a certain $\lambda > 0$. *Then we have* $\limsup_{\varepsilon \to 0} \mu_\varepsilon^{(k)} \leqq \mu^{(k)}$ *for each* $k \in N$.

Remark 2. Rauch and Taylor [6] discussed in detail various concrete domain perturbations for the Laplacian, which assure (2), although the domain perturbation of [5] is not treated by [6]; theorem 4.1 of L. Boccardo and P. Marcellini [3] also describes the asymptotic properties of eigenvalues of the Laplacian (cf. theorem 3.71 of H. Attouch [1]), but we can not apply this theorem to the perturbation of [5]. However, the same method as in [5] shows that the perturbation of [5] fills the assumption (2).

2. Proof of Theorem 1. Using a monotone theory, more specifically, the theory of subdifferentials (cf. H. Brezis [2]) and the Borsuk-Ulam theorem we prove theorem 1. We define a convex lower semicontinuous function $\varphi^\varepsilon: X \to [0, \infty]$ by (i) $\varphi^\varepsilon(x)=a_\varepsilon(x)/2$, $x \in V_\varepsilon$, (ii) $\varphi^\varepsilon(x)=\infty$, $x \in X\backslash V_\varepsilon$. Let $\partial\varphi^\varepsilon$ be the subdifferential of φ^ε. Then we have (iii) $\partial\varphi^\varepsilon(x)=A_\varepsilon x+H_\varepsilon^\perp$, $x \in D(\partial\varphi^\varepsilon)$, (iv) $(1+\lambda\partial\varphi^\varepsilon)^{-1}=(1+\lambda A_\varepsilon)^{-1}P_\varepsilon$, $\lambda>0$, where $D(\partial\varphi^\varepsilon)=D(A_\varepsilon)$ and $H_\varepsilon^\perp$ is the orthogonal complement of H_ε. We write $J_\lambda^\varepsilon=(1+\lambda\partial\varphi^\varepsilon)^{-1}$. Next we convert (1) to a min-max form. For a linear subspace M of X set $g^{(k)}(M) =\{F=\tilde{F}\cap\Sigma \mid \tilde{F}$ is a k dimensional linear subspace of $M\}$ and $g_+^{(k)}(M) = \cup\{g^{(m)}(M) \mid m\geqq k\}$. If $M=X$ we write $g^{(k)}$ and $g_+^{(k)}$ instead of $g^{(k)}(X)$ and $g_+^{(k)}(X)$, respectively. Then we have $\mu_\varepsilon^{(k)}=\inf\{\sup a_\varepsilon(x) \mid g^{(k)}(V_\varepsilon)\ni F\} =\inf\{\sup a_\varepsilon(x) \mid g_+^{(k)}(V_\varepsilon)\ni F\}$. Thus we have the lemma below.

Lemma 3. *$\mu_\varepsilon^{(k)}/2=\inf\{\sup\varphi^\varepsilon(x) \mid g_+^{(k)}\ni F\}$ for each k.*

We have the following lemma.

Lemma 4. *We assume* (2). *Then, for any $F\in g_+^{(k)}$, we have ε_F and $F_\varepsilon\in g_+^{(k)}$, $0<\varepsilon<\varepsilon_F$, such that*

$$\limsup_{\varepsilon\to 0}\sup_{F_\varepsilon\ni x}\varphi^\varepsilon(x)\leqq\sup_{F\ni x}\varphi(x). \tag{3}$$

Theorem 1 follows from lemmas 3, 4. Actually, by lemma 3 we have $F_n\in g_+^{(k)}$ such that $\mu^{(k)}/2=\lim_{n\to\infty}\sup\{\varphi(x) \mid F_n\ni x\}$. Thus we obtain $\mu_\varepsilon^{(k)}/2 \leqq\sup\{\varphi^\varepsilon(x) \mid F_{n,\varepsilon}\ni x\}\leqq\sup\{\varphi(x) \mid F_n\ni x\}+n^{-1}$, $0<\varepsilon<\varepsilon_n$ with $g_+^{(k)}\ni F_{n,\varepsilon}$, $\varepsilon_n\downarrow 0$ by lemmas 3, 4. Therefore theorem 1 is proved.

To see lemma 4 it suffices to prove the next lemma because of the Borsuk-Ulam theorem: If B is a bounded open symmetric neighborhood of 0 in $\boldsymbol{R}^m$ and T is an odd, continuous map from ∂B into a proper subspace of $\boldsymbol{R}^m$ then there is $x\in\partial B$ such that $Tx=0$.

Lemma 5. *We assume* (2). *Then, for any $F\in g_+^{(k)}$, there is a sequence of odd, continuous maps $T_\varepsilon: F\to\Sigma$ satisfying* (3) *with $F_\varepsilon=T_\varepsilon F$.*

To construct T_ε we recall properties of the Yosida approximation $\varphi_\lambda^\varepsilon$ of φ^ε: (v) $\varphi_\lambda^\varepsilon$ is of class C^1 on X and $(\varphi_\lambda^\varepsilon)'=\lambda^{-1}(1-J_\lambda^\varepsilon)$, $\lambda>0$ (we write $A_\lambda^\varepsilon =(\varphi_\lambda^\varepsilon)'$), (vi) A_λ^ε is Lipschitz continuous with constant λ^{-1}, (vii) $\varphi_\lambda^\varepsilon(x) =\lambda|A_\lambda^\varepsilon x|^2/2+\varphi^\varepsilon(J_\lambda^\varepsilon x)\leqq\varphi^\varepsilon(x)$ for all $x\in X$, (viii) $P_\varepsilon=\text{s-lim}_{\lambda\to 0}J_\lambda^\varepsilon$.

Proposition 6. *$\varphi^\varepsilon J_\lambda^\varepsilon\to\varphi J_\lambda$ uniformly on F as $\varepsilon\to 0$.*

Proof. Since $\varphi^\varepsilon(0)=0$, we have $\varphi_\lambda^\varepsilon(0)=0$. By (v) and (vii) we obtain

$$\varphi^\varepsilon J_\lambda^\varepsilon x=\int_0^1(A_\lambda^\varepsilon(tx), x)dt-\lambda|A_\lambda^\varepsilon x|^2/2. \tag{4}$$

The sequence $\{(A_\lambda^\varepsilon(tx), x)\}_\varepsilon$ is uniformly bounded on (0, 1) by (vi). The pointwise convergence of $\varphi^\varepsilon J_\lambda^\varepsilon$ follows from (2), (v) and the Lebesgue convergence theorem. By (vi) $\{\varphi^\varepsilon J_\lambda^\varepsilon\}_\varepsilon$ is uniformly bounded and equi-continuous on F. Thus the lemma follows from the Ascoli-Arzela theorem.

Since (vii), (viii) and proposition 6 hold, it is natural to set $T_\varepsilon x =|J_{\lambda_0}^\varepsilon x|^{-1}J_{\lambda_0}^\varepsilon x$ for $x\in F$ with sufficiently small λ_0, $0<\varepsilon<\varepsilon(\lambda_0, F)$. If this map T_ε is actually well defined then T_ε is odd, continuous; we have

$$\varphi^\varepsilon T_\varepsilon x \leqq \inf_{F \ni y} |J_\lambda^\varepsilon y|^{-2} \sup_{F \ni z} \varphi^\varepsilon J_\lambda^\varepsilon z$$

for all $x \in F$. For (3) with $F_\varepsilon = T_\varepsilon F$ and the well definedness of T_ε we need (ix) $\inf_{F \ni x} |J_\lambda x| \to 1$ as $\lambda \to 0$, (x) $\inf_{F \ni x} |J_\lambda^\varepsilon x| \to \inf_{F \ni x} |J_\lambda x|$ as $\varepsilon \to 0$ for each $\lambda > 0$. Both of (ix) and (x) follow from the next lemma, because J_λ^ε and J_λ are contractive.

Lemma 7. *If $U = \text{s-lim}\, U_n$ on X and U_n is Lipschitz continuous with constant c_0, where c_0 is independent of n. Then U_n converges to U uniformly on any compact set.*

Now we have lemma 5 and the proof of theorem 1 is completed.

References

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