67. *Quadratic Spline Interpolation on a Jordan Curve*

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1. **Summary.** The existence, uniqueness and convergence properties of quadratic splines interpolating to a given function $f(z(t))$ at an intermediate point of each subarc have been studied.

2. **Existence and uniqueness.** Let I be the interval $[0,1]=\{t: 0\leqslant t\leqslant 1\}$, $\Delta=\{t_0, t_1, \cdots, t_n\}$, $0=t_0<t_1<\cdots<t_n=1$ a subdivision of I and $I_j=[t_{j-1}, t_j]$, the j-th subinterval of I. Let $K=\{z(t): t\in I\}$, $z(0)=z(1)$, be a closed Jordan curve and $K_j=\{z(t): t\in I_j\}$ the j-th subarc of K corresponding to Δ. Let furthermore λ be a number $\in(0,1)$. Put $f(z(t))=F(t)$, $h_j=t_j-t_{j-1}$ and $\alpha_j=t_{j-1}+\lambda h_j$ for $j=1,2,\cdots,n$ so that $z(\alpha_j)\in K_j$. Considering $q_\Delta(t)\in C^1(I)$ with the interpolatory condition

$$q_\Delta(\alpha_j)=F(\alpha_j) \qquad j=1,2,\cdots,n, \tag{2.1}$$

we shall prove the following:

Theorem 2.1. *If $f(z(t))$, $t\in I$, be a given functiton on K, then there exists a unique periodic quadratic spline $q_\Delta(t)\in C^1(I)$ satisfying the interpolatory condition* (2.1).

Proof of Theorem 2.1. Let $P(t)=(t-t_j)(t-t_{j-1})(t-\alpha_j)$. We suppose that in I_j,

$$q_\Delta(t)=AP_j(t)-BP_{j-1}(t)-CP_j(t,\alpha) \tag{2.2}$$

where $P_i(t)$ $(i=j, j-1)$ is $P(t)$ without $(t-t_i)$ and $P_j(t,\alpha)$ is $P(t)$ without $(t-\alpha_j)$ (cf. [2]).

Writing $q'_\Delta(t_j)=M_j$, $j=1,2,\cdots,n$, and using (2.1) we have from (2.2)

$$M_j h_j^{-1}=(2-\lambda)A-(1-\lambda)B-F_j(\alpha,h;\lambda) \tag{2.3}$$

$$M_{j-1}h_j^{-1}=-\lambda A+(1+\lambda)B+F_j(\alpha,h;\lambda) \tag{2.4}$$

where

$$F_j(\alpha,h;\lambda)=\lambda^{-1}(1-\lambda)^{-1}h_j^{-2}F(\alpha_j). \tag{2.5}$$

Using (2.3)–(2.4), we get another expression for $q_\Delta(t)$:

$$\begin{aligned}2q_\Delta(t)=&M_{j-1}h_j^{-1}((1-\lambda)P_j(t)-(2-\lambda)P_{j-1}(t))\\&+M_jh_j^{-1}((1+\lambda)P_j(t)-\lambda P_{j-1}(t))\\&+2F_j(\alpha,h;\lambda)(\lambda P_j(t)+(1-\lambda)P_{j-1}(t)-P_j(t,\alpha)).\end{aligned} \tag{2.6}$$

Since $q_\Delta(t_j-)=q_\Delta(t_j+)$, $j=1,2,\cdots,n$; we get

$$\begin{aligned}(1-\lambda)^2a_jM_{j-1}+((1-\lambda^2)a_j+(2\lambda-\lambda^2)b_j)M_j+\lambda^2b_jM_{j+1}\\=2(h_j+h_{j+1})^{-1}(F(\alpha_{j+1})-F(\alpha_j))\end{aligned} \tag{2.7}$$

where

$$a_j=h_j/(h_j+h_{j+1}) \quad\text{and}\quad b_j=1-a_j. \tag{2.8}$$

The existence and uniqueness of the spline $q_\Delta(t)$ rests upon the existence of a unique solution of the equations (2.7) in M_j's. This follows if the coefficient matrix of the equations has dominant main diagonal. The coefficients of M_{j-1}, M_j and M_{j+1} in (2.7) are positive. Now the difference of the coefficient of M_j over the sum of the coefficients of M_{j-1} and M_{j+1} is $2\lambda(1-\lambda)$ which is positive. Hence the matrix of coefficients of M_j's in (2.7) becomes diagonally dominant and unique M_j's are determined. This completes the proof of Theorem 2.1.

Remark 2.1. We may represent the spline $q_\Delta(t)$ in terms of its value at the mesh points, $q_\Delta(t_j)=m_j$. Thus on I_j, we have

(2.9) $$h_j^2 q_\Delta(t)=m_{j-1}(\lambda^{-1}P_{j-1}(t))+m_j((1-\lambda)^{-1}P_j(t))-F_j(\alpha, h; \lambda)(h_j^2 P_j(t, \alpha)).$$

Since $q'_\Delta(t_j-)=q'_\Delta(t_j+)$ for $j=1, 2, \cdots, n$, we get

(2.10) $$(1-\lambda)^2 h_{j+1}m_{j-1}+((1-\lambda^2)h_j+(2\lambda-\lambda^2)h_{j+1})m_j+\lambda^2 h_j m_{j+1} \\ =h_jF(\alpha_{j+1})+h_{j+1}F(\alpha_j).$$

It is easy to see that elements of the matrix of this system are positive. Under the conditions of Theorem 2.1 we know that $q_\Delta(t)$ exists and is unique. Hence system (2.10) has a unique solution.

3. Convergence. It may be observed that the row max norm of the inverse of the coefficient matrix in (2.7) is less than or equal to $(2\lambda-2\lambda^2)^{-1}$ (cf. [1]). In the sequel $\omega(F; h)$ will denote the modulus of continuity of F. Set $e(t)=q_\Delta(t)-F(t)$ and $e_j^{(\nu)}=e^{(\nu)}(t_j)$, $\nu=0, 1, 2$. Considering $F\in C^2$ on I, we shall prove the following:

Theorem 3.1. *Let $F(t)$ be of class C^2 on I. Let $q_\Delta(t)\in C^1(I)$ be the periodic quadratic spline satisfying* (2.1). *Then for all t*

$$|q_\Delta^{(2)}(t)-F^{(2)}(t)|\leqslant(2MC_1+1)\omega(F''; \bar{\Delta})$$
$$|q_\Delta^{(\nu)}(t)-F^{(\nu)}(t)|\leqslant(2M+1/2)(\bar{\Delta})^{2-\nu}\omega(F''; \bar{\Delta}), \qquad \nu=0, 1$$

where

(3.1) $$\bar{\Delta}=\max_j h_j$$

(3.2) $$\max_j h_j\leqslant C_1 \min_j h_j.$$

Proof of Theorem 3.1. From the Eq. (2.7) after some simplifications, we can easily write the system of equations for $e_j^{(1)}$ as follows:

(3.3) $$(1-\lambda)^2 a_j e_{j-1}^{(1)}+(ab)_j(\lambda)e_j^{(1)}+\lambda^2 b_j e_{j+1}^{(1)}=U_j$$

where

(3.4) $$(ab)_j(\lambda)=(1-\lambda^2)a_j+(2\lambda-\lambda^2)b_j,$$

(3.5) $$U_j=(1-\lambda)^2 a_j h_j(F''(\eta_j)-F''(\xi_j))+\lambda^2 b_j h_{j+1}(F''(\xi_{j+1})-F''(\eta_{j+1})),$$

ξ_j, ξ_{j+1} are some points lying in (α_j, t_j) and (t_j, α_{j+1}) respectively and $\eta_i\in I_i$ for $i=j, j+1$.

Following the proof of Theorem 2 in [3]; we get

$$\max|e_j^{(1)}|\leqslant M\bar{\Delta}\omega(F''; \bar{\Delta})$$

where M is an appropriate positive constant. Next, by the reasoning in Kammerer, Reddien and Varga ([4], p. 245),

(3.6) $$e^{(2)}(t)=(e_j^{(1)}-e_{j-1}^{(1)})/h_j+F''(\tau)-F''(t)$$

from which it follows that

(3.7) $$|e^{(2)}(t)|\leqslant(1+2MC_1)\omega(F''; \bar{\Delta}).$$

To find a bound for $e^{(1)}(t)$, again, by an argument similar to that in

[4], we find

(3.8) $$|e^{(1)}(t)|\leqslant\bar{\Delta}(1/2+2M)\omega(F'';\bar{\Delta}).$$

The bound for e is obtained directly by integration.

4. Case when $F\in C^1(I)$.

Theorem 4.1. *Let $F(t)$ be of class C^1 on I. Let $q_\Delta(t)$ be the quadratic spline of Theorem* 3.1. *Then*

$$|q_\Delta^{(\nu)}(t)-F^{(\nu)}(t)|\leqslant N(\bar{\Delta})^{1-\nu}\omega(F';\bar{\Delta}),\qquad \nu=0,1.$$

The proof is based on the system of equations (2.7) and is parallel to the proof of theorem 4 in [3].

5. Case when $F\in C(I)$.

Theorem 5.1. *Let $F(t)\in C(I)$. Let $q_\Delta(t)$ be the quadratic spline of Theorem* 3.1. *Then*

$$|q_\Delta(t)-F(t)|\leqslant R\omega(F;\bar{\Delta}).$$

The proof is based on the system of equations (2.10) and follows the same lines as above with suitable modifications.

If $t_{r,j}$ $(r=1,2,\cdots;j=1,2,\cdots,n_r)$ is a sequence of subdivisions of [0, 1] and $\bar{\Delta}_r=\max(t_{r,j}-t_{r,j-1})$, then we have the following corollaries.

Corollary 5.1. Suppose $F(t)$ satisfies the conditions of Theorem 3.1, $\bar{\Delta}_r\to 0$ as $r\to\infty$ and (3.2) holds uniformly. Then the corresponding quadratic splines $q_{\Delta_r}^{(\nu)}(t)\to F^{(\nu)}(t)$, $\nu=0,1,2$ uniformly as $\bar{\Delta}_r\to 0$.

Corollary 5.2. Suppose $F(t)$ satisfies the conditions of Theorem 4.1 and $\bar{\Delta}_r\to 0$ as $r\to\infty$. Then corresponding quadratic splines $q_{\Delta_r}^{(\nu)}(t)\to F^{(\nu)}(t)$, $\nu=0,1$ uniformly as $\bar{\Delta}_r\to 0$.

Corollary 5.3. Let $F(t)\in C$ on K and $\bar{\Delta}_r\to 0$ as $r\to\infty$. Then $q_{\Delta_r}(t)\to F(t)$ uniformly as $\bar{\Delta}_r\to 0$.

References

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