65. A Construction of Lie Algebras and Lie Superalgebras by Freudenthal-Kantor Triple Systems. I

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1. **Introduction.** In our previous paper [3], we showed that from a two-dimensional associative triple system W and any generalized Jordan triple system $U(-1, 1)$ of second order (due to I. L. Kantor [4]) we can make a generalized Jordan triple system $W\otimes U(-1, 1)$ of second order which induces the Lie triple system, and that we have a Lie algebra as a standard embedding of the Lie triple system. In this paper, it is shown that Lie algebras and Lie superalgebras can be also constructed by Freudenthal-Kantor triple system $U(\varepsilon, \delta)$ $(\varepsilon=\pm 1, \delta=\pm 1)$ which becomes a generalized Jordan triple system of second order in case $\varepsilon=-1, \delta=1$. We can make, namely, from the same associative triple system W as in [3] and any Freudenthal-Kantor triple system $U(\varepsilon, \delta)$ a Freudenthal-Kantor triple system $W\otimes U(\varepsilon, \delta)$ to which we can associate a Lie algebra and a Lie superalgebra as a standard embedding of a Lie triple system $W\otimes U(\varepsilon, \delta)\oplus \overline{W\otimes U(\varepsilon, \delta)}$, where $\overline{W\otimes U(\varepsilon, \delta)}$ is an isomorphic copy of $W\otimes U(\varepsilon, \delta)$. We assume that any vector space considered in this paper is finite-dimensional and the characteristic of the base field Φ is different from 2 or 3. The author wishes to express his hearty thanks to Prof. K. Yamaguti for his kind advice and encouragement.

2. A triple system A with a trilinear product $\{abc\}$ is called an *associative triple system* (ATS) if $\{ab\{cde\}\}=\{a\{bcd\}e\}=\{\{abc\}de\}=\{a\{dcb\}e\}$ for all elements $a, b, c, d, e \in A$ [6].

Let W be a two-dimensional triple system which has a basis $\{e_1, e_2\}$ such that

$$(1)\qquad \begin{aligned}&\{e_1e_1e_1\}=\alpha e_1, \quad \{e_1e_1e_2\}=\{e_1e_2e_1\}=\{e_2e_1e_1\}=\alpha e_2,\\ &\{e_1e_2e_2\}=\{e_2e_1e_2\}=\{e_2e_2e_1\}=\beta e_1, \quad \{e_2e_2e_2\}=\beta e_2,\end{aligned}$$

where $\alpha, \beta \in \Phi$. Then W is a commutative ATS and is also a Jordan triple system.

In the ATS W, we have

$$(2)\qquad l(a, b)l(c, d)=l(c, d)l(a, b),$$

$$(3)\qquad l(a, b)l(c, d)=l(l(a, b)c, d)=l(c, l(b, a)d),$$

where $l(a, b)c=\{abc\}$, for $a, b, c, d \in W$.

A *Freudenthal-Kantor triple system* (FSK) $U(\varepsilon, \delta)$ is a vector space with a triple product $\{xyz\}$ satisfying

$$(4)\qquad [L(x, y), L(u, v)]=L(L(x, y)u, v)+\varepsilon L(u, L(y, x)v),$$

(5) $K(K(x, y)u, v)=L(v, u)K(x, y)-\varepsilon K(x, y)L(u, v)$,
where $L(x, y)u=\{xyu\}$ and $K(x, y)u=\{xuy\}-\delta\{yux\}$ for $x, y, u, v \in U(\varepsilon, \delta)$, $\varepsilon=\pm 1$, $\delta=\pm 1$.

Using (2) and (3), we have

Proposition. *For the ATS W and any FKS $U(\varepsilon, \delta)$, define a trilinear product in $W\otimes U(\varepsilon, \delta)$ by $\{a\otimes x\ b\otimes y\ c\otimes z\}=\{abc\}\otimes\{xyz\}$ for $a, b, c \in W$, $x, y, z \in U(\varepsilon, \delta)$. Then $W\otimes U(\varepsilon, \delta)$ becomes an FKS and $K(a\otimes x, b\otimes y)=l(a, b)\otimes K(x, y)$.*

Corollary. *If $U(\varepsilon, \delta)$ is a Jordan triple system (or an anti-Jordan triple system), then so is $W\otimes U(\varepsilon, \delta)$.*

A triple system $T(\delta)$ with product $[abc]$, $\delta=\pm 1$, is called a *Lie triple system* (LTS) if it satisfies the following identities for any elements $x, y, z, u, v \in T(\delta)$ ([9])

(i) $[xyz]=-\delta[yxz]$,

(ii) $[xyz]+[yzx]+[zxy]=0$,

(iii) $[xy[uvz]]=[[xyu]vz]+[u[xyv]z]+[uv[xyz]]$.

When $\delta=1$, $T(1)$ is an ordinary Lie triple system [5] and when $\delta=-1$, $T(-1)$ is an anti-Lie triple system [2].

For the FKS $W\otimes U(\varepsilon, \delta)$, we consider a vector space direct sum $W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)}$, where $\overline{W\otimes U(\varepsilon, \delta)}$ is an isomorphic copy of $W\otimes U(\varepsilon, \delta)$, of which element is denoted by a finite sum of the matrix form $\begin{pmatrix} a\otimes x \\ b\otimes y \end{pmatrix}$ and defined a triple product on it by

$$(6) \qquad \left[\begin{pmatrix} a_1\otimes x_1 \\ a_2\otimes x_2 \end{pmatrix}\begin{pmatrix} b_1\otimes y_1 \\ b_2\otimes y_2 \end{pmatrix}\begin{pmatrix} c_1\otimes z_1 \\ c_2\otimes z_2 \end{pmatrix}\right]:=L\left(\begin{pmatrix} a_1\otimes x_1 \\ a_2\otimes x_2 \end{pmatrix}, \begin{pmatrix} b_1\otimes y_1 \\ b_2\otimes y_2 \end{pmatrix}\right)\begin{pmatrix} c_1\otimes z_1 \\ c_2\otimes z_2 \end{pmatrix}$$

$$:=\begin{pmatrix} \{a_1\otimes x_1\ b_2\otimes y_2\ c_1\otimes z_1\}-\delta\{b_1\otimes y_1\ a_2\otimes x_2\ c_1\otimes z_1\}+\delta K(a_1\otimes x_1, b_1\otimes y_1)c_2\otimes z_2 \\ \varepsilon\{b_2\otimes y_2\ a_1\otimes x_1\ c_2\otimes z_2\}-\varepsilon\delta\{a_2\otimes x_2\ b_1\otimes y_1\ c_2\otimes z_2\}-\varepsilon K(a_2\otimes x_2, b_2\otimes y_2)c_1\otimes z_1 \end{pmatrix}$$

$$=\begin{pmatrix} l(a_1, b_2)\otimes L(x_1, y_2)-\delta l(b_1, a_2)\otimes L(y_1, x_2) & \delta l(a_1, b_1)\otimes K(x_1, y_1) \\ -\varepsilon l(a_2, b_2)\otimes K(x_2, y_2) & \varepsilon l(b_2, a_1)\otimes L(y_2, x_1)-\varepsilon\delta l(a_2, b_1)\otimes L(x_2, y_1) \end{pmatrix}\begin{pmatrix} c_1\otimes z_1 \\ c_2\otimes z_2 \end{pmatrix},$$

then $W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)}$ becomes an LTS with respect to this product (cf. [9]).

Hence we have the following

Theorem 1. *Let W be the ATS as above and $U(\varepsilon, \delta)$* be an *FKS. Then, from an FKS $W\otimes U(\varepsilon, \delta)$ we obtain an LTS $W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)}$ which induces the standard embedding Lie algebra ($\delta=1$) and Lie superalgebra ($\delta=-1$)*

$$\mathfrak{G}(\varepsilon, \delta)=\mathfrak{D}\oplus W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)},$$

where $\mathfrak{D}$ is the Lie algebra of inner derivations in the LTS $W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)}$.

Put $\mathfrak{G}=V_0\oplus V_1$, where $V_0=\mathfrak{D}$, $V_1=W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)}$, then we have $[V_0, V_0]\subset V_0$, $[V_0, V_1]\subset V_1$, $[V_1, V_1]\subset V_0$.

More precisely, let

$\mathfrak{G}_{-2}$ be the vector space spanned by derivations $L\left(\begin{pmatrix} a\otimes x \\ 0 \end{pmatrix}, \begin{pmatrix} b\otimes y \\ 0 \end{pmatrix}\right)$,

$\mathfrak{G}_0$ be the vector space spanned by derivations $L\left(\begin{pmatrix} a\otimes x \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b\otimes y \end{pmatrix}\right)$,

$\mathfrak{G}_2$ be the vector space spanned by derivations $L\left(\begin{pmatrix} 0 \\ a\otimes x \end{pmatrix}, \begin{pmatrix} 0 \\ b\otimes y \end{pmatrix}\right)$,

$\mathfrak{G}_{-1}=W\otimes U(\varepsilon, \delta)$ and $\mathfrak{G}_{+1}=\overline{W\otimes U(\varepsilon, \delta)}$,

where $a, b\in W$ and $x, y\in U(\varepsilon, \delta)$. Then $V_0=\mathfrak{G}_{-2}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_2$, $V_1=\mathfrak{G}_{-1}\oplus\mathfrak{G}_1$.

By straightforward calculations we have

Theorem 2. *The Lie algebra or Lie superalgebra obtained in Theorem 1 is the graded Lie algebra or Lie superalgebra of second order such that*

$$\mathfrak{G}(\varepsilon, \delta)=\mathfrak{G}_{-2}\oplus\mathfrak{G}_{-1}\oplus\mathfrak{G}_0\oplus\mathfrak{G}_1\oplus\mathfrak{G}_2,$$

$[\mathfrak{G}_i, \mathfrak{G}_j]\subset\mathfrak{G}_{i+j}$ *for* $i, j=0, \pm 1$ *and* ± 2 (*it being understood that* $\mathfrak{G}_{i+j}=0$ *if* $i+j$ *is different from* 0, ± 1 *and* ± 2) (cf. [8]).

3. Example. The complex number field $\boldsymbol{C}$ becomes an FKS $U(-1, 1)$ relative to the product $\{xyz\}=x(\bar{y}z)+(\bar{y}x)-y(\bar{x}z)$ (cf. [1, 4]). By direct calculations, we see that $\dim\mathfrak{G}_{-2}=\dim\mathfrak{G}_2=\dim\mathfrak{G}_0=2$ and $\dim\mathfrak{G}_{-1}=\dim\mathfrak{G}_1=4$, and LTS $W\otimes\boldsymbol{C}\oplus\overline{W\otimes\boldsymbol{C}}$ is simple. Hence $\dim\mathfrak{G}(W, \boldsymbol{C})=14$ and $\mathfrak{G}(W, \boldsymbol{C})$ is of type G_2.

Remark. K. Yamaguti has defined a bilinear form γ on an FKS $U(\varepsilon, \delta)$ by $\gamma(x, y)=(1/2)Sp[(\delta+1)(R(x, y)-\varepsilon R(y, x))+\varepsilon\delta L(x, y)-L(y, x)]$ where $R(x, y)z=\{zxy\}$ ([8]). Using this definition, we obtain the bilinear forms γ_W and γ_1 on W and $W\otimes U(\varepsilon, \delta)$ as follows: $\gamma_W(a, b)=Spl(a, b)$ and $\gamma_1(a\otimes x, b\otimes y)=\gamma_W(a, b)\gamma(x, y)$ respectively. And also T. S. Ravisankar has defined the Killing form on LTS by

$$\kappa(x, y)=(1/2)Sp[R(x, y)+R(y, x)],$$

where $R(x, y)z=[zxy]$ [7]. Using this definition, we have the Killing form κ on LTS $W\otimes U(\varepsilon, \delta)\oplus\overline{W\otimes U(\varepsilon, \delta)}$ as follows:

$$\kappa\left(\begin{pmatrix} a_1\otimes x_1 \\ a_2\otimes x_2 \end{pmatrix}, \begin{pmatrix} b_1\otimes y_1 \\ b_2\otimes y_2 \end{pmatrix}\right)=\gamma_W(a_1, b_2)\gamma(x_1, y_2)+\gamma_W(a_2, b_1)\gamma(x_2, y_1).$$

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