63. Construction of Certain Vector Valued Siegel Modular Forms of Degree two

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§1. **Introduction.** Let St be the standard representation of $GL(2, \boldsymbol{C})$ and $V(k, r)$ a representation space of $\det^k \otimes \mathrm{Sym}^r\,\mathrm{St}$. We denote the full Siegel modular group of degree two by Γ_2. A C^∞-Siegel modular form f of type (k, r) and of degree two is a $V(k, r)$ valued C^∞-function on the Siegel upper half plane H_2 of degree two satisfying the equation

$$f((AZ+B)(CZ+D)^{-1}) = (\det^k \otimes \mathrm{Sym}^r\,\mathrm{St})(CZ+D)f(Z)$$

$$\text{for all } M=\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$$

and the usual growth rate condition (see Borel [2, §7]). We denote by $M^\infty_{k,r}(\Gamma_2)$ the $\boldsymbol{C}$-vector space of all such functions. We put

$$M_{k,r}(\Gamma_2) = \{f \in M^\infty_{k,r}(\Gamma_2) \mid f \text{ is holomorphic on } H_2\}.$$

We shall explicitly construct $M_{k,2}(\Gamma_2)$ for even k and prove some congruences of eigenvalues of Hecke operators. Details of this paper are included in [8]. The author would like to thank Prof. R. Tsushima for communicating his paper [13] before publication and Prof. N. Kurokawa for his encouragement.

§2. **Construction of modular forms of type $(k, 2)$.** Let S_2 be the $\boldsymbol{C}$-vector space of complex symmetric matrices of size two. The representation of $GL(2, \boldsymbol{C})$ defined via $A \to \det(G)^k G A {}^tG$ for $G \in GL(2, \boldsymbol{C})$ and $A \in S_2$ is equivalent to $\det^k \otimes \mathrm{Sym}^2\,\mathrm{St}$. Henceforth, we put $V(k, 2) = S_2$. We denote by $M^\infty_k(\Gamma_n)$ the $\boldsymbol{C}$-vector space of C^∞-Siegel modular forms of degree n and weight k. Let $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ be subspaces of $M^\infty_k(\Gamma_n)$ consisting of holomorphic Siegel modular forms and of holomorphic cusp forms, respectively. We agree that $M_k(\Gamma_2) = \{0\}$ for a negative k. For a variable $Z = \begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}$ on H_2 we put

$$Y = \frac{1}{2i}(Z - \bar{Z}) \quad \text{and} \quad \frac{d}{dZ} = \begin{bmatrix} \dfrac{\partial}{\partial z_1} & \dfrac{1}{2}\cdot\dfrac{\partial}{\partial z_3} \\ \dfrac{1}{2}\cdot\dfrac{\partial}{\partial z_3} & \dfrac{\partial}{\partial z_2} \end{bmatrix}.$$

We define a differential operator $\nabla = \nabla_k$ acting on $M_k(\Gamma_2)$ by

$$\nabla f = \frac{k}{2\pi i}(2iY)^{-1} f + \frac{1}{2\pi i}\frac{d}{dZ} f.$$

By Shimura [11, (4.5)], we have $\nabla f \in M^\infty_{k,2}(\Gamma_2)$ for $f \in M_k(\Gamma_2)$. For $f \in M_k(\Gamma_2)$ and $g \in M_j(\Gamma_2)$, we put

$$[f, g]=\frac{1}{2\pi i}\left(\frac{1}{k} g \frac{d}{dZ} f-\frac{1}{j} f \frac{d}{dZ} g\right).$$

Then, we have $[f, g] \in M_{k+j,2}(\Gamma_2)$. We use usual notation for particular modular forms (see Resnikoff and Saldaña [7], Igusa [3] and Kurokawa [4]). The following theorem is our main result.

Theorem 1. *For an even integer $k \geqq 0$, we have (as a $\boldsymbol{C}$-vector space)*

$$\begin{aligned} M_{k,2}(\Gamma_2)=&M_{k-10}(\Gamma_2)[\varphi_4, \varphi_6] \oplus M_{k-14}(\Gamma_2)[\varphi_4, \chi_{10}] \\ &\oplus M_{k-16}(\Gamma_2)[\varphi_4, \chi_{12}] \oplus V_{k-16}(\Gamma_2)[\varphi_6, \chi_{10}] \\ &\oplus V_{k-18}(\Gamma_2)[\varphi_6, \chi_{12}] \oplus W_{k-22}(\Gamma_2)[\chi_{10}, \chi_{12}] \end{aligned} \tag{1}$$

where

$$V_k(\Gamma_2)=M_k(\Gamma_2) \cap \boldsymbol{C}[\varphi_6, \chi_{10}, \chi_{12}] \quad \text{and}$$
$$W_k(\Gamma_2)=M_k(\Gamma_2) \cap \boldsymbol{C}[\chi_{10}, \chi_{12}].$$

The proof is as follows: the inclusion $\supset$ is trivial. Using the dimension formula obtained by Tsushima[12, Theorem 4], we see that the right hand side of (1) spans the left hand side.

A modular form f is said to be an eigenform if f is a *non-zero* common eigen function of all Hecke operators. We denote by $\lambda(m, f)$ the eigenvalue of the m-th Hecke operator normalized as in Arakawa [1, p. 164]. We put $\boldsymbol{Q}(f)=\boldsymbol{Q}(\lambda(m, f) \mid m \geqq 1)$.

Corollary 2. *Let $f \in M_{k,2}(\Gamma_2)$ be an eigenform for an even integer $k \geqq 0$. Then, $\boldsymbol{Q}(f)$ is a totally real finite extension of $\boldsymbol{Q}$, and eigenvalues $\lambda(m, f)$ are algebraic integers.*

The next theorem is utilized for the proof of congruences (4) below.

Theorem 3. *Let $f \in M_{k,2}(\Gamma_2)$ for an even integer $k \geqq 0$. Then there exists a unique C^∞-modular form $D(f) \in M_{k+2}^\infty(\Gamma_2)$ satisfying the following conditions* (a) *and* (b).

(a) *With respect to the Petersson inner product, $D(f)$ is orthogonal to each holomorphic cusp form $g \in S_{k+2}(\Gamma_2)$.*

(b) *The function $H(f)$ defined by*

$$H(f)=D(f)-\frac{1}{2} \frac{1}{\det(2\pi Y)} \operatorname{Tr}(2\pi Y f)$$

is holomorphic on H_2 and has the Fourier expansion of the form

$$H(f)(Z)=\sum_{N>0} a(N, H(f)) \exp(2\pi i \operatorname{Tr}(NZ)),$$

where N runs over all positive definite semi-integral matrices of size two.

Moreover $D(f)$ is an eigenform if f is an eigenform.

§ 3. Congruence formulas. For a cusp form $f \in S_{k+r}(\Gamma_1)$, we denote by $[f]_r \in M_{k,r}(\Gamma_2)$ the Klingen type Eisenstein series attached to f. Note $[f]_2=E_{k,2}\left(Z, f, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)$ in the notation of Arakawa [1, (1.4)]. If f is an eigenform, we see that $[f]_2$ is characterized as a unique eigenform satisfying $(\Phi[f]_2)(z)=f(z)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Using Theorem 1 we see that an eigen basis of $M_{14,2}(\Gamma_2)$ is $\{[\Delta_{16}]_2, [\varphi_4, \chi_{10}]\}$. Then we have the following congruence formulas for all $m \geqq 1$:

(2) $\lambda(m, [\Delta_{16}]_2) \equiv \lambda(m, [\varphi_4, \chi_{10}]) \bmod 373,$

(3) $m\lambda(m, \chi_{14}) \equiv \lambda(m, [\varphi_4, \chi_{10}]) \bmod 5 \cdot 7,$

(4) $N_{K/Q}(m\lambda(m, [\varphi_4, \chi_{10}]) - \lambda(m, \chi_{16}^{(\pm)})) \equiv 0 \bmod 13,$

where $K = \boldsymbol{Q}(\sqrt{51349})$ and $N_{K/Q}$ is the norm map (cf. Kurokawa [4, § 3]). We note an interpretation concerning congruence (2) above. Let $f \in S_k(\Gamma_1)$ be an eigen form, $L_2(s, f)$ the second L-function attached to f and $\langle f, f \rangle$ its Petersson inner product normalized as in Shimura [10, (2.1)]. Put $L_2^*(s, f) = L_2(s, f)(2\pi)^{-(2s-k+2)}\Gamma(s)/\langle f, f \rangle$. Then, $L_2^*(s, f)$ belongs to $\boldsymbol{Q}(f)$ for each even integer s satisfying $k \leqq s \leqq 2k-2$ by Zagier [14, Theorem 2]. Numerical computation shows $373 \mid L_2^*(28, \Delta_{16})$. Here we note $28 = 2(k+r) - 2 - r$ with $k = 14$ and $r = 2$. More generally we expect that $L_2^*(2(k+r) - 2 - r, f)$ appears in the denominator of Fourier coefficients of $[f]_r$. The case $r = 0$ is proved in Mizumoto [6] (cf. Kurokawa [5]). On the other hand, congruences (3) and (4) correspond to the different weight case treated by Serre [9, Theorem 10, case (i)]. Hence, primes appearing in congruences of this type would be rather small.

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