62. A Note on the Mean Value of the Zeta and L-functions. I

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University

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1. The aim of the present series of notes is to develop a study on the various mean values of the Riemann zeta- and Dirichlet L-functions; here, to begin with, we investigate the square mean of L-functions viewing it as a generalization of the situation considered by Atkinson [1].

Let χ be a Dirichlet character, and put, for two complex variables u and v

$$Q(u, v; q)=\frac{1}{\varphi(q)}\sum_{\chi \pmod q} L(u, \chi)L(v, \bar{\chi}),$$

where $q\geqq 2$ and φ is the Euler function. If $Re(u)>1$, $Re(v)>1$, then

(1) $$Q(u, v, q)=L(u+v, \chi_0)+f(u, v; q)+f(v, u; q),$$

where χ_0 is the principal character mod q, and

$$f(u, v; q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(qm+a)^{-u}(q(m+n)+a)^{-v}.$$

We need an analytic continuation of $f(u, v; q)$ valid when $Re(u)<1$, $Re(v)<1$. This may be obtained by Poisson's summation formula as in [1], but we take an alternative way which starts from the following integral representation: When $Re(u)>0$, $Re(v)>1$, $Re(u+v)>2$,

$$f(u, v; q)=\frac{q^{-u-v}}{\Gamma(u)\Gamma(v)}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_0^\infty \frac{y^{v-1}}{e^y-1}\int_0^\infty \frac{e^{(a/q)(x+y)}}{e^{x+y}-1}x^{u-1}dxdy.$$

To remove the singularity at $x+y=0$ we put

$$h(z; q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\frac{e^{(a/q)z}}{e^z-1}-\frac{1}{z}\right),$$

and note that when $0<Re(u)<1$ and $y>0$

$$\int_0^\infty x^{u-1}(x+y)^{-1}dx=y^{u-1}\Gamma(u)\Gamma(1-u).$$

Then, we find that when $0<Re(u)<1$, $Re(u+v)>2$,

(2) $$f(u, v; q) =\varphi(q)q^{-(u+v)}\Gamma(u+v-1)\Gamma(1-u)\{\Gamma(v)\}^{-1}\zeta(u+v-1)+g(u, v; q),$$

where

$$g(u, v; q)=\frac{q^{-u-v}}{\Gamma(u)\Gamma(v)}\int_0^\infty \frac{y^{v-1}}{e^y-1}\int_0^\infty h(x+y; q)x^{u-1}dxdy.$$

Next we introduce the contour C which starts at infinity, proceeds along the positive real axis to δ $(0<\delta<1/2)$, describes a circle of radius δ counterclockwise round the origin and returns to infinity along the positive real axis; we have, for $0<Re(u)<1$, $Re(u+v)>2$,

(3) $g(u, v; q)$

$$=q^{-u-v}\{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)\}^{-1}\int_C \frac{y^{v-1}}{e^y-1}\int_C h(x+y; q)x^{u-1}dxdy,$$

where $x^u=\exp(u\log x)$, $y^v=\exp(v\log y)$ and $\mathrm{Im}\log x$, $\mathrm{Im}\log y$ vary from 0 to 2π round C. But this double integral is absolutely convergent for $Re(u)<1$ and arbitrary v; thus (2) and (3) provide $f(u, v; q)$ the required analytic continuation. Hence from (1)–(3) we see that when $Re(u)<1$, $Re(v)<1$,

$$Q(u, v; q)=L(u+v; \chi_0)+\varphi(q)q^{-u-v}\Gamma(u+v-1)\zeta(u+v-1)\cdot\left\{\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\right\}+g(u, v; q)+g(v, u; q).$$

In particular, setting $v=1-u$, we obtain

Lemma 1. *If* $0<Re(u)<1$, *then*

$$Q(u, 1-u; q)=\frac{\varphi(q)}{q}\left\{\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(u)+\frac{\Gamma'}{\Gamma}(1-u)\right)+2\gamma+\log\frac{q}{2\pi}+\sum_{p\mid q}\frac{\log p}{p-1}\right\}+g(u, 1-u; q)+g(1-u, u; q),$$

where γ is the Euler constant, and p runs over prime divisors of q; g-terms are defined by (3).

2. Now, as an application of the above result we consider the asymptotical estimation of

$$\frac{1}{\varphi(q)}\sum_{\chi \pmod q}\left|L\left(\frac{1}{2}+it, \chi\right)\right|^2,$$

where t is real. Heath-Brown [2] studied the special case where $t=0$, and obtained an expression which when q is a prime yields an asymptotic series in terms of $q^{-1/2}$. We consider this problem on a little more general condition that t be arbitrary but fixed. Lemma 1 reduces the problem to the estimation of $g(u, 1-u; q)$, $0<Re(u)<1$. For this sake we note first that

$$h(z; q)=\sum_{r\mid q}\mu\left(\frac{q}{r}\right)h\left(\frac{z}{r}; 1\right)$$

where μ is the Möbius function. Thus by (3) we get, after some rearrangement,

$$g(u, 1-u; q)=\frac{1}{q}\zeta(u)\zeta(1-u)\sum_{r\mid q}\mu\left(\frac{q}{r}\right)r^u$$

(4)
$$+\frac{1}{4\pi q\sin(\pi u)}\sum_{r\mid q}\mu\left(\frac{q}{r}\right)r^u\int_C\frac{y^{-u}}{e^y-1}\cdot\int_C\left(h\left(x+\frac{y}{r}; 1\right)-h(x; 1)\right)x^{u-1}dxdy.$$

This double integral admits an asymptotic expansion in terms of r^{-1} which arises from the power series expansion of $h(x+y/r; 1)-h(x; 1)$ in terms of y/r. But we are unable to proceed further without assuming that q has no small prime factors. Thus we restrict ourselves to the simplest situation where q is a prime number. Then (4) becomes

$$g(u,1-u;q)=q^{u-1}\zeta(u)\zeta(1-u)-q^{-1}g(u,1-u;1)$$
$$+\frac{q^u}{4\pi q\sin(\pi u)}\int_C\frac{y^{-u}}{e^y-1}\int_C\left(h\left(x+\frac{y}{q};1\right)-h(x;1)\right)x^{u-1}dxdy,$$

and this gives rise to an asymptotic expansion for $g(u,1-u;q)$. In particular we have

(5) $$g(u,1-u;q)=q^{u-1}\zeta(u)\zeta(1-u)-q^{-1}g(u,1-u;1)+O(|q^u|q^{-2}).$$

To show this we need only to remark that the differentiation gives

$$|h(x+(y/q);1)-h(x;1)|=O(q^{-1}|y|(1+|x|^2)^{-1})$$

uniformly for all $x, y\in C$. Thus by Lemma 1 and (5) we obtain

Theorem. *Let t be real and fixed, and let q run over prime numbers. Then we have*

$$(q-1)^{-1}\sum_{\chi\ (\mathrm{mod}\ q)}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2=\log\frac{q}{2\pi}+2\gamma+Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)$$
$$+2q^{-1/2}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\cos(t\log q)-q^{-1}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2+O(q^{-3/2}).$$

Remark. Our result agrees with that of Heath-Brown [2]; to see this one should note that $(\Gamma'/\Gamma)(1/2)=-\gamma-2\log 2$. Also it should be remarked that our result suggests some peculiar relation between the zeros of ζ and the values of L-functions.

3. The study of $Q(1/2+it, 1/2-it;q)$ for variable t and q, which is to be developed in our later notes, will naturally require more subtle analysis than that of the preceding paragraph. As a preparation we show here a further transformation of (3) when $u+v=1$, $Re(u)<0$:

Lemma 2. *If $Re(u)<0$, then*

$$g(u,1-u;q)=2q^{-1}\sum_{r|q}\mu\left(\frac{q}{r}\right)r\sum_{n=1}^{\infty}d(n)\int_0^{\infty}x^{-u}(x+1)^{u-1}\cos(2\pi rnx)dx,$$

where d is the divisor function.

This corresponds precisely to the expression of $g(u,1-u;1)$ shown in [1, p. 357]. As for the proof it may be enough to remark that when $Re(u)<0$ the inner integral of (3) is equal to minus the sum of all residues arising from the poles at $x=-y+2\pi in$ $(n=\pm 1,\pm 2,\cdots)$.

References

[1] F. V. Atkinson: The mean value of the Riemann zeta-function. Acta Math., **81**, 353–376 (1949).

[2] D. R. Heath-Brown: An asymptotic series for the mean value of Dirichlet L-functions. Comment. Math. Helv., **56**, 148–161 (1981).