# 59. On Whittaker Vectors for Generalized Gelfand-Graev Representations of Semisimple Lie Groups

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Let $G$ be a reductive algebraic group over a local (or a finite) field and $\mathfrak{g}$ its Lie algebra. A regular nilpotent element of $\mathfrak{g}$ gives canonically a non-degenerate character of a maximal unipotent subgroup. The representation of $G$ induced from such a character is called a *Gelfand-Graev representation*, and it is multiplicity free if $G$ is quasi-split. N. Kawanaka [3] generalized this construction using Dynkin's theory on nilpotent Ad $(G)$-orbits, and associated to every nilpotent orbit an induced representation called *generalized* Gelfand-Graev representation (GGGR). In [3], the GGGRs of finite reductive groups were studied in detail.

**1. Definition of GGGRs.** Let $G=KAN$ be an Iwasawa decomposition of a connected semisimple Lie group $G$ with finite center, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}\oplus\mathfrak{n}$ the corresponding decomposition of its Lie algebra $\mathfrak{g}$. Denote by $W$ the Weyl group of $(\mathfrak{g}, \mathfrak{a})$. Choose a positive system $\Lambda^+$ of the root system $\Lambda$ of $(\mathfrak{g}, \mathfrak{a})$ so that $\mathfrak{n}=\sum_{\lambda\in\Lambda^+}\mathfrak{g}_\lambda$, where $\mathfrak{g}_\lambda$ denotes the root space of $\lambda$. Let $U$ be the maximal unipotent subgroup with Lie algebra $\mathfrak{u}=\sum_{\lambda\in\Lambda^+}\mathfrak{g}_{-\lambda}$.

For a $C^\infty$-manifold $\Omega$ and a Fréchet space $E$, let $C^\infty(\Omega, E)$ (resp. $C_0^\infty(\Omega, E)$) denote the space of $E$-valued smooth functions on $\Omega$ (resp. those with compact supports) equipped with the Schwartz topology. Let $V$ be a closed subgroup of $G$ and $\eta$ a smooth representation (see e.g. [4, p. 254]) of $V$ on a Fréchet space $E$. The left translation defines a smooth representation $\pi_\eta$ of $G$ on the space $C_\eta^\infty(G, E)$ of $f$ in $C^\infty(G, E)$ satisfying $f(gv)=\eta(v)^{-1}f(g)$ $(g\in G,\ v\in V)$, which is equipped with the topology inherited from that of $C^\infty(G, E)$.

For a non-zero nilpotent element $X\in\mathfrak{g}$, by Jacobson-Morozov theorem, there exists an $\mathfrak{sl}_2$-triplet $\{X, H, Y\}\subset\mathfrak{g}$ containing $X$: $[H, X]=2X$, $[H, Y]=-2Y$, $[X, Y]=H$. By taking a suitable Ad $(G)$-conjugate of $X$, we may assume that $-H$ is dominant in $\mathfrak{a}$. Since $-\lambda(H)=0$, 1 or 2 for any simple root $\lambda$, we get a gradation $\mathfrak{g}=\sum_{i\in Z}\mathfrak{g}(i)$ by ad $(H)$. For $i\geqq 1$, $\mathfrak{u}(i)=\sum_{k\geqq i}\mathfrak{g}(k)$ is a Lie subalgebra of $\mathfrak{u}$. Since $\mathfrak{g}(i)$ and $\mathfrak{g}(j)$ are orthogonal with respect to the Killing form $B$ of $\mathfrak{g}$ if $i+j\neq 0$, there exists a subalgebra $\mathfrak{u}(1.5)$ of $\mathfrak{u}(1)$ which has following two properties: (i) $\mathfrak{u}(2)\subseteqq\mathfrak{u}(1.5)$ and $2\dim\mathfrak{u}(1.5)=\dim\mathfrak{u}(1)+\dim\mathfrak{u}(2)$, (ii) $B(Y, [\mathfrak{u}(1.5), \mathfrak{u}(1.5)])=(0)$. Then we can define a unitary character $\eta_X$ of $U(1.5)=\exp\mathfrak{u}(1.5)$ by $\eta_X(\exp Z)=\exp\sqrt{-1}B(Y, Z)$ for $Z\in\mathfrak{u}(1.5)$.

**Definition.** For a non-zero nilpotent element $X\in\mathfrak{g}$, the smooth representation $(\pi_{\eta_X}, C^\infty_{\eta_X}(G, C))$ is called a *generalized Gelfand-Graev representation* (GGGR) associated to $X$.

The group $U(1.5)$ is not uniquely determined in general. Nevertheless we call every representation as above a GGGR.

Take a subset $F$ of the set $\Pi$ of simple roots in $\Delta^+$. Let $P_F=M_FA_FN_F$ $(A_F\subseteqq A, N_F\subseteqq N)$ be a Langlands decomposition of the standard parabolic subgroup $P_F$ corresponding to $F$, where $F$ generates the restricted root system of $M_F$. Let $U_F$ be the connected subgroup of $U$ opposite to $N_F$ and $U(F)=U\cap M_F$.

For a smooth representation $(\sigma, E_\sigma)$ of $M_F$ and $\nu\in(\mathfrak{a}_F)^*_C$, the complexified dual space of $\mathfrak{a}_F=\mathrm{Lie}(A_F)$, $\xi=\sigma\otimes e^{\nu+\rho}\otimes(1_{N_F})$ defines a smooth representation of $P_F$, where $\rho(Z)=2^{-1}\operatorname{tr}(\operatorname{ad}(Z)|\mathfrak{n})$ for $Z\in\mathfrak{a}_F$. Put $(\pi_{\sigma,\nu}, H_{\sigma,\nu})$ $=(\pi_\xi, C^\infty_\xi(G, E_\sigma))$. In the following, we treat the spaces $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_X})$ of continuous intertwining operators from the principal series representations $\pi_{\sigma,\nu}$ to the GGGRs $\pi_{\eta_X}$ for various $(F, \xi)$ and $X$.

**2. Uniqueness of intertwining operators.** We estimate $\dim \mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_X})$ using Bruhat's method. Let $\eta$ be a character of a closed subgroup $U'$ of $U$. We put

$\mathrm{Wh}_\eta(H^\vee_{\sigma,\nu})=\{T\in H^\vee_{\sigma,\nu}; \langle T, \pi_{\sigma,\nu}(u)f\rangle=\eta(u)\langle T, f\rangle$ for $u\in U', f\in H_{\sigma,\nu}\}$,
where $E^\vee$ denotes the dual space of a topological vector space $E$ and $\langle\,,\rangle$ the canonical inner product of $E^\vee$ and $E$. Each element in $\mathrm{Wh}_\eta(H^\vee_{\sigma,\nu})$ is called a *Whittaker vector* of type $(U', \eta)$.

Let $\mathcal{T}_{\eta,\sigma,\nu}$ be the space of $E_\sigma$-distributions $T$ on $G$ satisfying

(2.1) $\langle T, L_uR_{p^{-1}}\phi\rangle=\langle T, \eta(u)a^{\nu-\rho}\sigma(m)\phi\rangle (u\in U', p=man\in M_FA_FN_F)$

for $\phi\in C^\infty_0(G, E_\sigma)$, where $L_y\phi(x)=\phi(y^{-1}x)$, $R_y\phi(x)=\phi(xy)$ $(x, y\in G)$.

For an $s\in W$, take a representative $s^*$ of $s$ in $K$ and put $G_s=Us^*P_F$. Then $G=\coprod_{s\in W/W_F}G_s$ (Bruhat decomposition), where $W_F$ denotes the subgroup of $W$ generated by reflections corresponding to elements of $F$. Let $\Omega_s$ be the union of $G_s$ with $G_{s'}$ of strictly larger dimension. Then $\Omega_s$ is an open subset of $G$ and $G_s$ is a closed submanifold of $\Omega_s$. Let $\mathcal{T}^s_{\eta,\sigma,\nu}$ denote the space of $E_\sigma$-distributions $T$ on $\Omega_s$ which satisfy the condition (2.1) for $\phi\in C^\infty_0(\Omega_s, E_\sigma)$ and have supports contained in $G_s$.

**Proposition 1.** *It holds that* $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_\eta)\simeq \mathrm{Wh}_\eta(H^\vee_{\sigma,\nu})\simeq\mathcal{T}_{\eta,\sigma,\nu}$ *(as vector spaces), and* $\dim\mathcal{T}_{\eta,\sigma,\nu}\leqq\sum_{s\in W/W_F}\dim\mathcal{T}^s_{\eta,\sigma,\nu}$.

Now we assume that $\eta$ be a character of $U_{F'}$ for some $F'\subseteqq\Pi$. Suggested by Prop. 1, we study the spaces $\mathcal{T}^s_{\eta,\sigma,\nu}$. We estimate the support of $T\in\mathcal{T}^s_{\eta,\sigma,\nu}$ as follows.

**Theorem 2.** *Assume that $\sigma$ is finite dimensional. For every $s$ in $W$, a distribution in $\mathcal{T}^s_{\eta,\sigma,\nu}$ has always its support contained in $D^s_\eta U_{F'}s^*P_F$, where $D^s_\eta=\{y\in U(F')\cap s^*U_Fs^{*-1}; \eta|U_{F'}\cap ys^*P_F(ys^*)^{-1}=1\}$.*

We apply Th. 2 to $\eta=\eta_X$. For linear groups $G$, we can show that $D^s_{\eta_X}$ $=\phi$ if $F=F'$ and $s\notin W_F$. So we get the following theorem on a uniqueness property of Whittaker vectors for GGGRs.

**Theorem 3.** *Assume that G be a linear group. Let X be a non-zero* *nilpotent element with* $U(1.5)=U_{F'}$ *for some* $F'\subseteqq \Pi$. *For a finite dimensional representation* $(\sigma, E_\sigma)$ *of* $M_F$ *and* $\nu\in(\mathfrak{a}_F)_C^*$, *one has*

(i) $\mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_X})=(0)$ *if* $\mathrm{Ad}(G)Y\cap \mathfrak{n}_F=\phi$,

(ii) $\dim \mathrm{Hom}_G(\pi_{\sigma,\nu}, \pi_{\eta_X})\leqq \dim E_\sigma$ *if* $F=F'$; $=0$ *if* $F\supsetneqq F'$.

**Remark.** The assumption "$U(1.5)=U_{F'}$ for some $F'\subseteqq\Pi$" for $X$ is satisfied for any even nilpotent element $X$, and also for any nilpotent $X$ if $\mathfrak{g}$ is a complex simple Lie algebra of type $A_l$.

**3. Whittaker integrals and intertwining operators.** We construct Whittaker vectors for GGGRs through Whittaker integrals and their analytic continuation. Suggested by Casselman's subrepresentation theorem, we consider the representations $\pi_{\sigma,\nu}$ induced from the minimal parabolic subgroup $P=MAN$. Let $(\sigma, E_\sigma)$ be an irreducible finite dimensional representation of $M$ and $\nu\in\mathfrak{a}_C^*$. For an $s\in W$, put $U_s=U\cap s^{*-1}Ns^*$. Then $\{U_s; s\in W\}\supseteqq\{U_F; F\subseteqq\Pi\}$. For a unitary character $\eta$ of $U_s$, we introduce a *Whittaker integral*

(3.1) $$W^{e^\vee}(\sigma,\nu,\eta)f(g)=\int_{U_s}\langle e^\vee, f(gu)\rangle\eta(u)du \quad (g\in G, f\in H_{\sigma,\nu})$$

for $e^\vee\in E_\sigma^\vee\backslash(0)$, where $du$ denotes a Haar measure on $U_s$.

Define an open convex tubular domain $D_s$ in $\mathfrak{a}_C^*$ by $D_s=\{\nu\in\mathfrak{a}_C^*; \langle \mathrm{Re}\,\nu, \lambda\rangle >0$ for $\lambda\in\langle\!\langle s\rangle\!\rangle\}$, where $\langle\!\langle s\rangle\!\rangle$ denotes the set of positive roots $\lambda$ such that $s\lambda<0$. The following proposition is a slight generalization of [1, Prop. 2.4].

**Proposition 4.** *The integral* (3.1) *is absolutely convergent for* $\nu\in D_s$. *Moreover* $W^{e^\vee}(\sigma,\nu,\eta)f(g)$ *is smooth in* $g\in G$ *and holomorphic with respect to* $\nu\in D_s$. *The map* $f\to W^{e^\vee}(\sigma,\nu,\eta)f$ *gives a non-zero intertwining operator from* $H_{\sigma,\nu}$ *to* $C_\eta^\infty(G, \boldsymbol{C})$.

To construct intertwining operators for general $\nu\in\mathfrak{a}_C^*$, we consider analytic continuation of Whittaker integrals and examine it in detail. From now on, we assume that $G$ is defined over $\boldsymbol{C}$ for a technical reason. For $w\in W$ such $U_w\subseteqq U_s$, put $W'_{\eta,w}=W_{F(\eta,w)}$ with $F(\eta,w)=\{\lambda\in\Pi\cap\langle\!\langle sw^{-1}\rangle\!\rangle; \eta|\exp(\mathfrak{g}_{-w^{-1}\lambda})\neq 1\}$. Combining Jacquet's result [2, p. 277] with the analytic continuation of intertwining operators between two principal series representations, we get the following

**Proposition 5.** *Let* $w$ *be as above and assume that* $\eta|U_w=1$. *Then* $W^{e^\vee}(\sigma,\nu,\eta)f(g)$ *extends to a meromorphic function of* $\nu$ *in* $w^{-1}[W'_{\eta,w}D_{sw^{-1}}]$ *for every K-finite vector* $f\in H_{\sigma,\nu}$, *where* $[\omega]$ *denotes the convex hull of a set* $\omega\subseteqq\mathfrak{a}_C^*$.

In case $\eta=\eta_X$, we examine the existence of $w\in W$ satisfying the assumption of Prop. 5 and $[W'_{\eta,w}D_{sw^{-1}}]=\mathfrak{a}_C^*$ and prove it for certain nilpotent elements including all for type $A_l$ as follows.

**Theorem 6.** *Let* $\mathfrak{g}=\bigoplus_j\mathfrak{g}^j$ *be the direct sum decomposition of a complex semisimple Lie algebra* $\mathfrak{g}$ *into simple ideals* $\mathfrak{g}^j$. *For* $Z\in\mathfrak{g}$, *write* $Z=\sum_j Z^j$ *with* $Z^j\in\mathfrak{g}^j$. *Let* $X_0$ *be a non-zero nilpotent element in* $\mathfrak{g}(\Pi)=\sum_{\lambda\in\Pi}\mathfrak{g}_{-\lambda}$ *such that* $X_0^j$ *is even unless* $\mathfrak{g}^j$ *is of type* $A_l$. *Then there exists a* $w\in W$ *such that* $X=\mathrm{Ad}(w^*)X_0$ *satisfies the following conditions.* (i)

*There exists a subset $F(X)$ of $\Pi$ such that $U(1.5)$ can be taken as $U_{F(X)}$.*
(ii) *The function $W^{e^\vee}(\sigma, \nu, \eta_X)f(g)$ extends to a meromorphic function of $\nu$ on the whole $\mathfrak{a}_C^*$ for every K-finite vector $f$ in $H_{\sigma,\nu}$.*

This theorem generalizes, in complex case, Jacquet's result [2, p. 280] for regular nilpotent elements, and assures the existence of infinitesimal Whittaker vectors for general $\nu \in \mathfrak{a}_C^*$.

In case of type $A_l$, the Whittaker integral extends meromorphically to the whole $\mathfrak{a}_C^*$ for every $X$ by Th. 6.

**Example** ($\mathfrak{g}$ of rank 2). We can take $U(1.5)$ as $U_s$ for every $X$. Using Prop. 5, we can show that $W^{e^\vee}(\sigma, \nu, \eta_X)f(g)$ extends to a meromorphic function on $\mathfrak{a}_C^*$ except only one case for type $G_2$. In this exceptional case, the weighted Dynkin diagram of the orbit is given as $0 \circ\!\Lleftarrow\!\circ\, 2$, and it never intersects $\mathfrak{g}(\Pi)$. The Whittaker integral extends meromorphically to a half space by Prop. 5, but we don't know if it extends meromorphically to a larger domain or not.

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