57. On a Theorem of R. H. Martin on Certain Cauchy Problems for Ordinary Differential Equations*)

By Giovanni EMMANUELE
Department of Mathematics, University of Catania, Catania, Italy

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1. Introduction. Let E be a real Banach space with norm $\|\cdot\|$ and X be a locally closed and convex subset of E. If $B, C: [0, 1]\times X\to E$ are two (suitable) functions, we consider the following Cauchy problem

(CP) $\dot{x}=B(t, x)+C(t, x), \quad x(0)=x_0$

where $x_0\in X$.

In the paper [2] R. H. Martin obtained the existence of a local solution of (CP) under the following assumptions;

(C_1) B and C are continuous and bounded in $[0, 1]\times X$;

(C_2) $\langle x-y, B(t, x)-B(t, y)\rangle\leq\omega(t, \|x-y\|)\|x-y\|$ for all (t, x), (t, y) in $[0, 1]\times X$, where $\omega(t, u)$: $[0, 1]\times[0, \infty)\to[0, \infty)$ is a continuous function such that $\omega(t, 0)=0$ for all $t\in[0, 1]$ and for which the Cauchy problem $\dot{u}=\omega(t, u)$, $u(0)=0$ has the unique solution $u(t)=0$ for all $t\in[0, 1]$;

(C_3) K is a relatively compact subset of E such that $C(t, x)\in K$ for all $(t, x)\in[0, 1]\times X$;

(C_4) $\liminf_{h\to+0} d(x+h(B(t, x)+C(t, x)); X)/h=0$ for all $(t, x)\in[0, 1]\times X$;

(C_5) C is uniformly continuous on $[0, 1]\times X$.

A diligent examination of the proof of this result shows the important role of the assumptions (C_3) and (C_5).

The hypothesis (C_3) plays a fundamental role also in other results contained in the same paper of Martin; however, recently (see [1]) it has been weakened using the following one;

(C_3)′ there is a Lebesgue measurable subset J of $[0, 1]$ with Lebesgue measure $m(J)=0$ for which $C(t, X)$ is relatively compact for any $t\in J^c$ (J^c denotes the complement of J in $[0, 1]$)

in the setting of Gelfand-Phillips spaces, so improving certain results of [2].

Purpose of this note is to generalize the above cited result of Martin in general Banach spaces using (C_3)′ instead of (C_3).

2. The existence results. This section contains the announced generalization of Martin's theorem. Together (C_3)′ we shall also use the following other assumptions;

(C_1)′ $B+C$ is continuous on $[0, 1]\times X$ and B and C are both bounded

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(instead of (C_1));

$(C_2)'$ the same as in (C_2) with ω now satisfying the following assumptions of Carathéodory type: ω is measurable in $t\in[0,1]$ for any $u\in[0,\infty)$, continuous in $u\in[0,\infty)$ for almost all $t\in[0,1]$; moreover, there exists $\alpha\in L^1([0,1])$ such that $\omega(t,u)\le\alpha(t)$ for all $(t,u)\in[0,1]\times[0,\infty)$;

$(C_5)'$ for each $\varepsilon>0$ there is a Lebesgue measurable subset I_ε of $[0,1]$ with Lebesgue measure $m(I_\varepsilon)<\varepsilon$ such that $C(t,x)$ is uniformly continuous on $I_\varepsilon^c\times X$ (instead of (C_5)).

Before proving our result we make a brief remark on the assumption $(C_5)'$; it is not difficult to construct examples of functions C verifying $(C_5)'$ but not (C_5) starting from the following fact: if $h:[0,1]\to E$ is a non-continuous element of $L^1(m,E)$, then for any $\varepsilon>0$ there is a Lebesgue measurable subset A_ε of $[0,1]$ such that i) $m(A_\varepsilon)<\varepsilon$; ii) $A_\varepsilon^c=[0,1]\backslash A_\varepsilon$ is compact; iii) h is continuous on A_ε^c (this is Lusin's theorem); hence we have that h is uniformly continuous on A_ε^c.

Now, we are ready to show our

Theorem. *Let B and C be two functions from $[0,1]\times X$ into E verifying the assumptions $(C_1)'$, $(C_2)'$, $(C_3)'$, (C_4) and $(C_5)'$. Then, the Cauchy problem* (CP) *has a local solution.*

Proof. Using $(C_1)'$ and (C_4) we can construct (see [3]) a subinterval $[0,T]$ of $[0,1]$, a real sequence (ε_n) converging to zero and an equicontinuous sequence (x_n) of absolutely continuous functions from $[0,T]$ into X such that: j) $x_n(0)=x_0$ for any $n\in N$; jj) $\|x_n(t)-x_n(s)\|\le M|t-s|$ for any $t,s\in[0,T]$ and any $n\in N$ (here M is a real number such that $\|B(t,x)\|+\|C(t,x)\|\le M$ for all $(t,x)\in[0,1]\times X$); jjj) for any $n\in N$ and almost all $t\in[0,T]$ one has $\|\dot{x}_n(t)-[B(t,x_n(t))+C(t,x_n(t))]\|\le\varepsilon_n$.

If we shall prove that (x_n) has a subsequence which converges uniformly on $[0,T]$, its limit function will be a solution of (CP) (with a standard proof).

Given $\sigma>0$ we choose a Lebesgue measurable subset J_σ of $[0,T]$ with $J_\sigma\supset[0,T]\cap J$ (J is like in $(C_3)'$) with $m(J_\sigma)<\sigma/12M$; moreover in $(C_5)'$ we take a Lebesgue measurable subset of $[0,1]$ with $m(I_\sigma)<\sigma/12M$; hence, $m(J_\sigma\cup I_\sigma)<\sigma/6M$; moreover, regularity of Lebesgue measure allows us to suppose that J_σ and I_σ are open in $[0,1]$; hence $A_\sigma=J_\sigma\cup I_\sigma$ is open in $[0,1]$; this implies that $A_\sigma^c\cap[0,T]$ is closed in $[0,T]$ (for brevity we suppose that $A_\sigma^c\subset[0,T]$) and moreover

(a) $C(t,X)$ is relatively compact for any $t\in A_\sigma^c$

(b) $C(t,x)$ is uniformly continuous on $A_\sigma^c\times X$.

Now, we consider the functions $y_n(t)=C(t,x_n(t))$, $t\in[0,T]$, $n\in N$; from (b) it follows that (y_n) is an equicontinuous (bounded) sequence of $C^0(A_\sigma^c;E)$; since using (a) we have that $\{y_n(t):n\in N\}$ is relatively compact in E for any $t\in A_\sigma^c$ the Ascoli Arzelà Theorem allows us to conclude that (y_n) is relatively compact; for brevity, we shall suppose that the same (y_n) converges uniformly on A_σ^c. Now, we shall prove that (x_n) is uniformly converging on $[0, T]$. If we put $h_{nm}(t)=\|x_n(t)-x_m(t)\|$, as in [2], we can write the following inequality which is true almost everywhere in $[0, T]$

$$\begin{aligned}(d/dt)(h^2_{nm}(t)) &\leq 2(x_n(t)-x_m(t),\ \dot{x}_n(t)-\dot{x}_m(t))\\ &\leq 2h_{nm}(t)[\omega(t,\ h_{nm}(t))+\|y_n(t)-y_m(t)\|+\varepsilon_n+\varepsilon_m]\\ &\leq 2h_{nm}(t)\omega(t,\ h_{nm}(t))+K\|y_n(t)-y_m(t)\|+K(\varepsilon_n+\varepsilon_m)\end{aligned}$$

for any $n, m \in N$ where $K=4(MT+\|x_0\|)$; hence, one has for $r, s \in [0, T]$, $s<r$, and any $n, m \in N$

$$\begin{aligned}h^2_{nm}(r)-h^2_{nm}(s) &\leq \int_s^r 2h_{nm}(\tau)\omega(\tau,\ h_{nm}(\tau))d\tau+TK(\varepsilon_n+\varepsilon_m)+\int_0^T K\|y_n(\tau)-y_m(\tau)\|\,d\tau\\ &=\int_s^r 2h_{nm}(\tau)\omega(\tau,\ h_{nm}(\tau))d\tau+TK(\varepsilon_n+\varepsilon_m)\\ &\quad +K\int_{A_\sigma}^{0}\|y_n(\tau)-y_m(\tau)\|\,d\tau+K\int_{A^c_\sigma}\|y_n(\tau)-y_m(\tau)\|\,d\tau\,;\end{aligned}$$

if n, m are sufficiently large we have that $T(\varepsilon_n+\varepsilon_m)<\sigma/3$ and further $\int_{A^c_\sigma}\|y_n(\tau)-y_m(\tau)\|\,d\tau<\sigma/3$; moreover, we have $\int_{A_\sigma}\|y_n(\tau)-y_m(\tau)\|\,d\tau<m(A_\sigma)2M<\sigma/3$; this signifies that for n, m sufficiently large and for $r, s \in [0, T]$ we have

$$h^2_{nm}(r)-h^2_{nm}(s)\leq \int_s^r 2h_{nm}(\tau)\omega(\tau,\ h_{nm}(\tau))d\tau+K\sigma.$$

Now, we suppose that (x_n) does not converge; hence, there exist an $\eta>0$ and two real sequences (n_ν) and (m_ν) such that $\|x_{n_\nu}-x_{m_\nu}\|_{C^0([0,T];E)}>\eta$; if we consider the sequence $(h_{n_\nu m_\nu})$ of real continuous functions we can write easily

$$k^2_\nu(r)-k^2_\nu(s)\leq \int_s^r 2k_\nu(\tau)\omega(\tau,\ k_\nu(\tau))\,d\tau+K\sigma$$

for any $r, s \in [0, T]$ and any ν sufficiently large, where $k_\nu=h_{n_\nu m_\nu}$ for any $\nu \in N$, for the sake of brevity.

Now, we observe that (k_ν) is a (bounded) sequence of equicontinuous functions of $C^0([0, T])$; hence, it has a uniformly converging subsequence to a continuous function k: $[0, T]\to R$; from the last inequality easily follows

$$k^2(r)-k^2(s)\leq \int_s^r 2k(\tau)\omega(\tau,\ k(\tau))d\tau$$

for any $r, s \in [0, T]$, using also the arbitrarity of σ. Moreover, if we recall the definition of any k_ν, we can easily conclude that k is an absolutely continuous function and hence it is differentiable almost everywhere on $[0, T]$; this allows to say that

$$k(t)\dot{k}(t)\leq \omega(t,\ k(t))k(t)$$

for almost all $t \in [0, T]$.

Now, we recall that $\|x_{n_\nu}-x_{m_\nu}\|_{C^0([0,T];E)}>\eta$ for any $\nu \in N$; this signifies that k cannot be identically null on $[0, T]$; hence, let $t^* \in]0, T[$ be a point such that $k(t^*)>0$. Let (α, β) be the maximal interval containing t^* such that $k(t)>0$ in $]\alpha, \beta[$ and $k(\alpha)=0$; we can define an absolutely continuous function h from $[0, T]$ into R by putting

$$h(t)=0\ (0\leqq t\leqq \alpha);\quad k(t)\ (\alpha<t<\beta);\quad k(\beta)\ (\beta\leqq t\leqq T).$$

Obviously, $\dot{h}(t)\leq\omega(t, h(t))$ almost everywhere on $[0, T]$; hence, $h(t)=0$ on $[0, T]$, i.e. a contradiction. Our proof is complete.

Remark 1. If we suppose that X contains a closed ball centered at x_0 we can drop the assumptions $(C_1)'$ and (C_4) which we used only in order to construct approximate solutions for (CP) and we can substitute them with the following one: *$B+C$ verifies assumptions of Carathéodory type.*

At the end, we observe that another recent improvement of Martin's result is due to Volkmann ([4]) who dropped the assumption (C_5), when X contains a closed ball centered at x_0; the new and interesting technique used in [4] seems however to require the assumptions (C_1) and (C_3), whereas our (usual) argument allows to use assumptions of Carathéodory type, when X contains a closed ball centered at x_0 as in [4] (see Remark 1); furthermore, we can improve (C_3) with $(C_3)'$ and this does not seem possible in [4]; hence, under this point of view our present result uses more large assumptions than that in [4]. But, we have to require the additional assumption $(C_5)'$ in order to prove our Theorem.

These facts imply that the present result is not actually comparable with that due to Volkmann.

References

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