54. A New Formulation of Local Boundary Value Problem in the Framework of Hyperfunctions. III

Propagation of Micro-Analyticity up to the Boundary

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(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1985)

In our previous notes ([4] and [5]) we have formulated boundary value problems in a unified way both for systems of linear partial differential equations with non-characteristic boundary and for single equations with regular singularities. In [5] we have also microlocalized this formulation. As an application of this microlocal formulation we present here some new results on propagation of micro-analyticity of solutions up to the boundary for equations satisfying a kind of micro-hyperbolicity; we can treat a class of equations for which the boundary is totally characteristic as well.

First let us briefly recall some of the definitions and results of [4] and [5]. Put

$$M=\boldsymbol{R}^n \ni x=(x_1, x'),\quad X=\boldsymbol{C}^n \ni z=(z_1, z'),\quad z'=(z_2, \cdots, z_n),$$
$$N=\{x \in M\,;\, x_1=0\},\quad Y=\{z \in X\,;\, z_1=0\},\quad \tilde{M}=\boldsymbol{R}\times\boldsymbol{C}^{n-1},$$
$$M_+=\{x \in M\,;\, x_1\geqq 0\},\quad \tilde{M}_+=\{(x_1, z') \in \tilde{M}\,;\, x_1\geqq 0\}.$$

We set $\mathcal{B}_{N|M_+}=(\iota_*\iota^{-1}\mathcal{B}_M)|_N$, where $\mathcal{B}_M$ is the sheaf of hyperfunctions on M and ι: $\operatorname{int} M_+\to M$ is the natural embedding. We use the notation $D=(D_1, D')$, $D'=(D_2, \cdots, D_n)$ with $D_j=\partial/\partial z_j$. We have defined in [5] a sheaf $\mathcal{C}_{M_+}$ on $S^*_M\tilde{M}$ and put $\mathcal{C}_{N|M_+}=\mathcal{C}_{M_+}|_{L_0}$ with $L_0=S^*_M\tilde{M}|_N\cong S^*_NY$. There is an exact sequence

$$0\longrightarrow\tilde{\iota}_*\tilde{\iota}^{-1}\mathcal{BO}|_N\longrightarrow\mathcal{B}_{N|M_+}\longrightarrow(\pi_{N/Y})_*\mathcal{C}_{N|M_+}\longrightarrow 0,$$

where $\tilde{\iota}$: $\operatorname{int}\tilde{M}_+\longrightarrow\tilde{M}$ is the embedding, $\pi_{N/Y}$: $S^*_NY\to N$ is the projection, $\mathcal{BO}$ is the sheaf on $\tilde{M}$ of hyperfunctions with holomorphic parameters z'.

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module (i.e. a system of linear partial differential equations with analytic coefficients) defined on a neighborhood in X of $\mathring{x}=(0, \mathring{x}') \in N$. First we assume

(N.C) Y is non-characteristic for $\mathcal{M}$.

Then there exist injective sheaf homomorphisms

$$\gamma:\quad \mathcal{H}om_{D_X}(\mathcal{M}, \mathcal{B}_{N|M_+})\longrightarrow\mathcal{H}om_{D_Y}(\mathcal{M}_Y, \mathcal{B}_N),$$
$$\gamma:\quad \mathcal{H}om_{D_X}(\mathcal{M}, \mathcal{C}_{N|M_+})\longrightarrow\mathcal{H}om_{D_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

compatible with each other, where $\mathcal{M}_Y$ is the tangential system of $\mathcal{M}$ to Y (Corollary of [4] and Theorem 3 of [5]).

Next let us assume

(R.S) $\mathcal{M}=\mathcal{D}_X/\mathcal{D}_XP$ with $P=a(x)\,((z_1D_1)^m+A_1(z, D')\,(z_1D_1)^{m-1}+\cdots+A_m(z, D'))$; here $a(z)$ is a holomorphic function with $a(\mathring{x})\neq 0$,

$A_j(z, D')$ is a linear partial differential operator of order $\leqq j$ with holomorphic coefficients such that $A_j(0, z', D')$ equals a function $a_j(z')$ for any j.

Moreover assume that $\lambda_i - \lambda_j \notin \boldsymbol{Z}\backslash\{0\}$ for any i, j; here λ_i are the roots of the equation

$$\lambda^m + a_1(\mathring{x})\lambda^{m-1} + \cdots + a_m(\mathring{x}) = 0.$$

Then there exist injective sheaf homomorphisms

$$\gamma: \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow (\mathcal{B}_N)^m,$$
$$\gamma: \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M_+}) \longrightarrow (\mathcal{C}_N)^m$$

on a neighborhood of $\mathring{x}$ (or of $(\pi_{N/Y})^{-1}(\mathring{x})$) compatible with each other (Theorems 2 and 3 of [5]).

Let H: $T^*(T^*X) \overset{\sim}{\to} T(T^*X)$ be the Hamilton map defined by $\langle \theta, v \rangle = \langle d\omega, v \wedge H(\theta) \rangle$ for $v \in T(T^*X)$, $\theta \in T^*(T^*X)$ with $\omega = \sum_{j=1}^n \zeta_j dz_j$, where ζ is the dual variable of z. Set $\theta_0 = dz_1$. Then $H(\theta_0)$ belongs to $T(T^*X|_{\tilde{M}})$. For a point x^* and subsets S, V of T^*X we denote by $C_{x^*}(S; V)$ the normal cone of S along V at x^* after Kashiwara-Schapira (Definition 1.1.1 of [2]). Note that $C_{x^*}(S; V)$ is a closed cone of the tangent space $T_{x^*}(T^*X)$ of T^*X at x^*. We denote by $\mathcal{E}_X$ the sheaf on T^*X of microdifferential operators of finite order.

Definition. A coherent $\mathcal{E}_X$-module $\mathcal{M}$ defined on a neighborhood of $x^* \in T^*_M X|_N$ in T^*X is called *micro-hyperbolic relative to* int $\tilde{M}_+$ *in the direction* θ_0 *at* x^* if and only if

$$H(\theta_0) \notin C_{x^*}(\mathrm{Supp}(\mathcal{M}) \cap T^*X|_{\mathrm{int}\,\tilde{M}_+}; T^*_M X).$$

Remark. (i) The condition above is equivalent to the following: there exist an open neighborhood U of x^* in $T^*X|_{\tilde{M}}$ and an open cone Γ in $T_{x^*}(T^*X|_{\tilde{M}}) \cong \{(x_1, z'; \zeta) \in \boldsymbol{R} \times \boldsymbol{C}^{n-1} \times \boldsymbol{C}^n\}$ containing $(0, 0; -1, 0, \cdots, 0)$ such that

$$((U \cap T^*_M X) + \Gamma) \cap U \cap \mathrm{Supp}(\mathcal{M}) \cap T^*X|_{\mathrm{int}\,\tilde{M}_+} = \phi.$$

(ii) If $\theta_0 \in T^*_{x^*}(T^*X)$ is micro-hyperbolic for $\mathcal{M}$ in the sense of [2], then $\mathcal{M}$ satisfies the condition above.

We identify $S^*_M X$ with $T^*_M X \backslash 0$ and denote by π: $T^*X \to X$ and ρ: $T^*X|_Y \to T^*Y$ the canonical projections. Note that $\mathcal{C}_{M_+}$ is supported by $L_0 \cup L_+$ with $L_+ = (\pi_{M/\tilde{M}})^{-1}(\mathrm{int}\, M_+)$.

Theorem 1. *Let x^* be a point of $S^*_N Y$ and $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a neighborhood of $\pi_{N/Y}(x^*)$. Assume the following conditions:*

(C.1) $$T^*_Y X \cap \mathrm{cl}(\mathrm{SS}(\mathcal{M}) \cap T^*X|_{\mathrm{int}\,\tilde{M}_+}) = \phi,$$

where cl *denotes the closure in T^*X, and* SS$(\mathcal{M})$ *denotes the characteristic variety of $\mathcal{M}$.*

(C.2) *$\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}$ is micro-hyperbolic relative to* int $\tilde{M}_+$ *in the direction θ_0 at each point of $\rho^{-1}(x^*) \cap T^*_M X$.*

(C.3) $\rho^{-1}(x^*) \cap \mathrm{cl}(\mathrm{SS}(\mathcal{M}) \cap T^*X|_{\mathrm{int}\,\tilde{M}_+}) \subset \{(\zeta_1, x^*) \in \rho^{-1}(x^*); \mathrm{Re}\,\zeta_1 \geqq 0\}$.

Under these conditions we have

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_{L_0}(\mathcal{C}_{M_+}))_{x^*} = 0.$$

There is a natural homomorphism

$$\psi: \ p^{-1}(\mathcal{C}_{M_+}|_{L_+}) \longrightarrow \mathcal{C}_M|_{p^{-1}(L_+)},$$

where $\mathcal{C}_M$ is the sheaf on S_M^*X of microfunctions and p is the natural projection of $S_M^*X \backslash S_{\tilde{M}}^*X$ to $S_M^*\tilde{M}$ (cf. [5]).

Theorem 2. *Let $\mathcal{M}$ satisfy* (N.C). *Suppose moreover that $\mathcal{M}$ satisfies* (C.2) *at $x^* \in S_N^*Y$ with $\mathring{x} = (\pi_{N/Y})(x^*)$ and*

(C.3)′ $\rho^{-1}(x^*) \cap \mathrm{SS}(\mathcal{M}) \subset \{(\zeta_1, x^*) \in \rho^{-1}(x^*)\,;\, \mathrm{Re}\,\zeta_1 \geqq 0\}$.

Under these assumptions, if f is a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M_+})$ at x^* such that $\psi(f)$ vanishes on $p^{-1}(U \cap L_+)$ with some neighborhood U of x^*, then $\gamma(f)$ vanishes as a germ of $\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$ at x^*.

Remark. This is a generalization to systems of a theorem of Kaneko [1] for single equations (see also Schapira [6], Kataoka [3], Sjöstrand [7]). However we believe that the following result for equations with regular singularities is essentially new.

Theorem 3. *Let $\mathcal{M}$ satisfy* (R.S) *and $\lambda_i - \lambda_j \notin \mathbf{Z} \backslash \{0\}$ for any i, j and let f be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M_+})$ at $x^* \in (\pi_{N/Y})^{-1}(\mathring{x})$. Assume moreover that there exists a neighborhood U of x^* in $S_M^*\tilde{M}$ such that $\sigma(P)(x, \zeta_1, \sqrt{-1}\xi') \neq 0$ for any $(x, \sqrt{-1}\xi') \in U \cap L_+$ and $\zeta_1 \in \mathbf{C}$ with* $\mathrm{Re}\,\zeta_1 < 0$ *and that $\psi(f)$ vanishes on $p^{-1}(U \cap L_+)$, where σ denotes the principal symbol. Then $\gamma(f)$ vanishes as a germ of $(\mathcal{C}_N)^m$ at x^*.*

Finally let us give a sketch of the proof of theorems: Theorem 1 is proved by the argument of prolongation of cohomology groups due to Kashiwara-Schapira [2]; we modify their argument and apply to cohomology groups with $\mathcal{B}\mathcal{O}$ coefficients instead of $\mathcal{O}_X$. Theorem 2 is an immediate consequence of Theorem 1 and the following:

Lemma. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module defined on a neighborhood of $\mathring{x}$ satisfying* (C. 1). *Then there exists a neighborhood V of $\mathring{x}$ in M such that the homomorphism*

$$\psi: \ p^{-1}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{M_+})|_{L_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)|_{p^{-1}(L_+)}$$

is injective on $p^{-1}(L_+ \cap (\pi_{M/\tilde{M}})^{-1}(V))$.

To prove Theorem 3 we use a coordinate transformation of the form $z_1 = w_1^k$ with an integer $k \geqq m$. Then we can apply Theorem 1 and Lemma by virtue of the local version of Bochner's tube theorem. Details of these arguments will appear elsewhere.

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