51. On the Positive Solutions of an Emden-Type Elliptic Equation

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§1. **Problem and main results.** Recently, many authors have been studying the problem of existence and uniqueness of positive entire solutions for second order semilinear elliptic equations. In this note we investigate the existence of positive entire solutions for the boundary value problem,

$$(\mathrm{E})\quad \begin{cases} (1.1) & \Delta u+\phi(|x|)u^m=0, \qquad x\in R^n, \\ (1.2) & u(x)\to b \qquad \text{as } |x|\to\infty, \end{cases}$$

where $n\geq 3$, $\Delta=\sum_{i=1}^n \partial^2/\partial x_i^2$, $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$ and b is a given nonnegative constant. We assume that $\phi(r)$ satisfies

$$(\mathrm{H})\quad \phi\in C(R_+),\quad \phi\geq 0 \text{ on } R_+,\quad \phi\not\equiv 0 \text{ and } \int_0^\infty r\phi(r)dr<\infty.$$

Here and hereafter, R_+ denotes the interval $[0,\infty)$.

It has already been shown by Kawano [1] that under the condition (H), (1.1) has infinitely many positive entire solutions. A similar result has also been obtained by Ni [2] under slightly stronger condition. Here we study the boundary value problem (E). Naito [3] and Fukagai [4] have investigated several subjects related to this problem. Our main results are as follows:

Theorem 1. *Let* $m>1$. *Suppose* (H) *and*

(H_1) $m<(n+2+2l)/(n-2)$, *where* $\phi(r)=c_l r^l+o(r^l)$ *at* $r=0$ *for some* $l\geq 0$ *and* $c_l>0$,

are satisfied. Then there exists some $B>0$ *such that*

(i) *for any* $b\in(B,\infty)$, (E) *has not any radially symmetric positive solution,*

(ii) *for any* $b\in[0,B]$, (E) *has a radially symmetric positive solution.*

Theorem 2. *Let* $m<1$. *Suppose* (H) *is satisfied. Then* (E) *has a radially symmetric positive solution for any* $b\in[0,\infty)$.

In the case of $m>1$, (H_1) seems to be crucial for the existence of positive solutions of (E) with $b=0$. The following is an example which breaks the condition (H_1) and always has strictly positive solutions, if we restrict ourselves to radially symmetric solution;

$$(1.3)\qquad \Delta u+\phi(|x|)u^5=0, \qquad x\in R^3,$$

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where $\phi(|x|)=6(1-|x|)$ for $|x|\leq 1$ and $\phi(|x|)=0$ for $|x|>1$. The fact is assured by using Pohozaev's technique.

§2. **Reduction to initial value problem.** Since we restrict ourselves to the positive solutions, the problem (E) is equivalent to

$$(\mathrm{E}')\quad \begin{cases} (2.1) & \Delta u+\phi(|x|)f(u)=0, \qquad x\in R^n, \\ (2.2) & u(x)\to b \qquad \text{as } |x|\to\infty, \end{cases}$$

where $f(u)$ is defined by $f(u)=u^m$ for $u\geq 0$ and $f(u)=0$ for $u<0$. As an auxiliary to (E′), we introduce an ODE initial value problem;

$$(\mathrm{E}_\alpha)\quad \begin{cases} (2.3) & v''(r)+(n-1)v'(r)/r+\phi(r)f(v(r))=0, \qquad r>0, \\ (2.4) & v(0)=\alpha, \qquad v'(0)=0, \end{cases}$$

where α is a positive parameter. It is easily shown that (E_α) is reduced to an integral equation,

$$v(r)=\alpha-(n-2)^{-1}\int_0^r \{1-(s/r)^{n-2}\}s\phi(s)f(v(s))ds. \tag{2.5}$$

By making use of this integral equation, we can prove the uniqueness and existence of the local C^2-solution of (E_α). For $m\geq 0$, the local solution is continued to R_+ by virtue of the modification of nonlinear term. For $m<0$, however, the solution is not naturally continued to R_+ if it goes to zero. Hereafter we denote the global unique solution of (E_α) by $v=v(r;\alpha)$.

§3. **Proof of Theorem 1.** Let $r_p=\inf\{r\in R_+;\ \phi(r)>0\}$ and $r_*=r_*(\alpha)$ $=\sup\{r\in R_+;\ v(r;\alpha)>0\}$. Then the following hold:

(i) $0\leq r_p<r_*<\infty$,

(ii) v is nonincreasing in $r\in R_+$,

(iii) v is bounded as $k(\alpha)\leq v\leq\alpha$, where $k(\alpha)=\alpha-\alpha^m(n-2)^{-1}\int_0^\infty s\phi(s)ds$,

(iv) v is strictly decreasing in $r\in[r_p,\infty)$.

Thus there exists $v_*(\alpha)=\lim_{r\to\infty} v(r;\alpha)$. Furthermore, we see the following:

(i) $v_*(\alpha)>0 \Longleftrightarrow v>0$ on R_+ and $\lim_{r\to\infty} v>0$,

(ii) $v_*(\alpha)=0 \Longleftrightarrow v>0$ on R_+ and $\lim_{r\to\infty} v=0$,

(iii) $v_*(\alpha)<0 \Longleftrightarrow \lim_{r\to\infty} v<0$.

Therefore it is sufficient for the present purpose to show that $\{v_*(\alpha);\ \alpha>0\}\cap R_+=[0,B]$ for some $B>0$.

It is easily seen from (2.5), (H) and Gronwall's inequality that $v_*(\alpha)$ is continuous in $\alpha>0$, $v_*(\alpha)>0$ for any sufficiently small $\alpha>0$ and $v_*(\alpha)\to 0$ as $\alpha\to 0$. Following the argument in [5, Proposition 3.9.] and using (H_1), we can show that $v_*(\alpha)<0$ for any sufficiently large α. This completes the proof.

If a weaker assumption,

(H_1') $m<(n+2+2l)/(n-2)$, where $\phi(r)=O(r^l)$ at $r=0$,

is introduced instead of (H_1), then it is shown by a result in [6] that $\{v_*(\alpha);\ \alpha>0\}\cap R_+\supset[0,B]$ for some $B>0$. The order l in (H_1') should be considered to be zero if $\phi(0)>0$, and infinity if there exists an r_0 such that $\phi(r)=0$ for

$r \in [0, r_0]$.

§4. **Proof of Theorem 2.** For $0 \leq m < 1$, it holds that $\{v_*(\alpha); \alpha > 0\} \cap R_+ = R_+$. In fact it is shown by (2.5), (H) and Ascoli-Arzelà's theorem that $v_*(\alpha)$ is continuous in α, $v_*(\alpha) < 0$ for sufficiently small α and $v_*(\alpha) \to \infty$ as $\alpha \to \infty$.

The method used for $m \geq 0$ is not available for $m < 0$, since $v_*(\alpha)$ is not defined for all $\alpha > 0$. In this case there exists a unique pair $\{r_d, v\} = \{r_d(\alpha), v(r;\alpha)\}$ such that $0 < r_d \leq \infty$, v is a solution of (E_α) on $[0, r_d)$, and $v \to 0$ as $r \to r_d$ if $r_d < \infty$. We see that v is nonincreasing in $r \in [0, r_d)$ and strictly decreasing in $r \in [r_p, r_d)$. We also see that v is bounded as $0 < v \leq \alpha$ and there exists a limiting value $v_*(\alpha) = \lim_{r \to \infty} v \geq 0$ if $r_d = \infty$.

It is shown by using (2.5) and (H) that

(i) $r_d < \infty$ for sufficiently small α,

(ii) if $r_d(\alpha) < \infty$, then $r_d(\alpha') < \infty$ for any α' sufficiently near to α,

(iii) $r_d = \infty$ for sufficiently large α, and $v_* \to \infty$ as $\alpha \to \infty$,

(iv) if $r_d(\alpha) = \infty$ and $v_*(\alpha) > 0$, then $r_d(\alpha') = \infty$ for any α' sufficiently near to α, and $v_*(\alpha') \to v_*(\alpha)$ as $\alpha' \to \alpha$.

Thus we have $\{v_*(\alpha); \alpha \in [\alpha_*, \infty)\} = R_+$, where $\alpha_* = \sup\{\alpha; r_d(\alpha) < \infty\}$. This completes the proof.

§5. **Concluding remark.** We have so far considered the existence of radially symmetric solutions for the Emden-type elliptic equations. However, the behavior of the auxiliary function $v_*(\alpha)$ introduced in the proof of theorems suggests the structure of solutions for a more general boundary value problem,

$$(\mathrm{P}) \quad \begin{cases} (5.1) \quad \Delta u + f(x, u) = 0, & x \in R^n, \\ (5.2) \quad u(x) \to b & \text{as } |x| \to \infty, \end{cases}$$

which includes (E) as a special case.

For instance, let a smooth function $f(x, u)$ satisfy that $f(x, u)$ is monotonically decreasing in u, $\lim_{u \downarrow 0} f(x, u) = +\infty$ and $f(x, u) > 0$ $(u > 0)$ for any fixed x. Then the existence of a solution of (P) for some $b > 0$ implies the unique existence of solutions of (P) for all $b \geq 0$. This result corresponds to the case $m < 0$ of the problem (E). Similar results corresponding to $m > 0$ can also be obtained. Concerning these points, we will discuss in a forthcoming paper [7].

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References

[1] N. Kawano: On bounded entire solutions of semilinear elliptic equations. Hiroshima Math. J., **14**, 125–158 (1984).

[2] W.-M. Ni: On the elliptic equation $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, its generalization, and applications in geometry. Indiana Univ. Math. J., **31**, 493–529 (1982).

[3] M. Naito: A note on bounded positive entire solutions of semilinear elliptic equations. Hiroshima Math. J., **14**, 211–214 (1984).

[4] N. Fukagai: Existence and uniqueness of entire solutions of second order sublinear elliptic equations (to appear in Funkcial Ekvac.).
[5] A. Haraux and F. B. Weissler: Non-uniqueness for a semilinear initial value problem. Indiana Univ. Math. J., **31**, 167–189 (1982).
[6] W.-M. Ni: A nonlinear Dirichlet problem on the unit ball and its applications. ibid., **31**, 801–807 (1982).
[7] N. Kawano, H. Matano, J. Satsuma and S. Yotsutani: (in preparation).