50. *Remarks on a Closed Subalgebra of a Banach Function Algebra*

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(Communicated by Kôsaku YOSIDA, M. J. A., June 11, 1985)

1. Let X be a compact Hausdorff space. We say that A is a *Banach function algebra* on X if A is a unital subalgebra of $C(X)$ with a Banach algebra norm which separates the points of X. A *function algebra* is a Banach function algebra with the uniform norm as the Banach algebra norm. It is well-known that $\|f\|_\infty \leq N(f)$ for all f in a Banach function algebra on X with the norm $N(\cdot)$, where $\|\cdot\|_\infty$ denotes the uniform norm on X. Some time ago I. Glicksberg [3] extended a theorem of Hoffman-Wermer [4] when X is metrizable. J. Wada [7] generalized the result of Glicksberg for the case that X is a compact Hausdorff space. He in fact showed the following:

Theorem W. *Let A be a function algebra on a compact Hausdorff space X. Let N be a closed linear subspace of $C(X)$ and I be a closed ideal in A with $A+\bar{I}\supset N\supset I$. If $N+\bar{I}$ is uniformly closed, then $I=\bar{I}$, where $\bar{I}$ denotes the complex conjugate of I,* i.e., $\bar{I}=\{f\in C(X):\bar{f}\in I\}$.

R. D. Mehta and M. H. Vasavada [5] showed a Wada's type theorem for the case of a Banach function algebra with the hypothesis of continuity for $f\mapsto\bar{f}$ on $A\cap\bar{A}$.

In this paper we obtain similar results concerning to a closed subalgebra of a Banach function algebra. As a corollary of the main result we show a Wada's type theorem for the case of a Banach function algebra which removes the hypothesis of a theorem of R. D. Mehta and M. H. Vasavada.

We denote *clE* the uniform closure of E for a subset E of $C(X)$ and $\bar{E}$ the complex conjugate of E, i.e., $\bar{E}=\{f\in C(X):\bar{f}\in E\}$. $A_R=A\cap C_R(X)$ is the set of all real valued continuous functions in A.

Let A be a Banach function algebra on a compact Hausdorff space X and I be a closed subalgebra of A. We define an equivalence relation $\approx$ in X as follows:

$$x\approx y \Longleftrightarrow f(x)=f(y) \qquad \text{for every } f \text{ in } I.$$

We denote $\{E_\alpha\}$ the equivalence class for the equivalence relation $\approx$, especially we denote $E_0=\{x\in X: f(x)=0 \text{ for } {}^\forall f\in I\}$. Our main result is the following:

Theorem. *Let A be a Banach function algebra on a compact Hausdorff space X. Let I be a closed subalgebra of A such that $I\cdot A_R\subset I$, where*

$I\cdot A_R=\{fg\in C(X): f\in I,\ g\in A_R\}$. *Suppose that* $A+\bar{I}\supset cl(I+\bar{I})$. *Then we have* $\bar{I}=I=\{f\in C(X): f=constant\ on\ E_\alpha\ for\ {}^\forall\alpha,\ f=0\ on\ E_0\}$.

2. We prove Theorem and show some corollaries in this section. By a theorem of Saeki [6, Theorem 3.3], it is easy to see:

Lemma 1. *Under the assumption of Theorem, we have* $clI=cl\bar{\bar{I}}=cl\bar{I}$.

We denote X/I the quotient space which is defined by identifying the points of X which cannot be separated by I. We may suppose that I is a subalgebra of $C(X/I)$. Let e be the point in X/I which corresponds to E_0.

Lemma 2. $clI=\{f\in C(X): f=constant\ on\ E_\alpha\ for\ {}^\forall\alpha,\ f=0\ on\ E_0\}$.

Proof. We suppose that $E_0=\phi$ for the first case. Let $[I]$ be a uniformly closed algebra generated by I and constant functions. We may regard $[I]$ as a function algebra on X/I, in fact we see that $[I]=C(X/I)$ by the Stone-Weierstrass theorem and Lemma 1. Thus we have

$$C(X/I)\cdot clI\subset clI$$

since clI is a closed subalgebra of $C(X/I)$, where

$$C(X/I)\cdot clI=\{fg\in C(X/I): f\in C(X/I),\ g\in clI\}.$$

There exists a finite number of functions $f_1, f_2, \cdots, f_n$ in I such that

$$|f_1|^2+|f_2|^2+\cdots+|f_n|^2>1/2$$

on X, for we suppose that $E_0=\phi$. So we have $1\in clI$, since

$$(|f_1|^2+|f_2|^2+\cdots+|f_n|^2)^{-1}\in C(X/I) \quad\text{and}\quad f_1\bar{f}_1+f_2\bar{f}_2+\cdots+f_n\bar{f}_n$$

is in $clI=cl\bar{I}$. Thus we get

$$clI=C(X/I)$$

or more precisely we have $clI=\{f\in C(X): f=\text{constant on } E_\alpha \text{ for } {}^\forall\alpha\}$ if $E_0=\phi$.

For the second case we suppose that $E_0\neq\phi$. By the same way as the first case, we see that $[I]=C(X/I)$ and that $C(X/I)\cdot clI\subset clI$. We may suppose that $X/I\sim\{e\}$ is a locally compact Hausdorff space, so we may suppose that clI is a closed subalgebra of $C_0(X/I\sim\{e\})$, where $C_0(X/I\sim\{e\})$ is the algebra of all complex valued bounded continuous functions on $X/I\sim\{e\}$ which vanish at infinity. Let Y be a compact subset of $X/I\sim\{e\}$. Then there are a finite number of functions $f_1, f_2, \cdots, f_n$ in I such that

$$|f_1|^2+|f_2|^2+\cdots+|f_n|^2>1/2$$

on Y. For each g in $C(Y)$, there is a G_j in $C_0(X/I\sim\{e\})$ for $j=1,2,\cdots,n$ such that

$$G_j|Y=g\cdot\bar{f}_j\cdot(|f_1|^2+|f_2|^2+\cdots+|f_n|^2)^{-1}.$$

We see that

$$(G_1f_1+G_2f_2+\cdots+G_nf_n)/Y=g$$

is in $(clI)|Y$ since $C_0(X/I\sim\{e\})\cdot clI\subset clI$, so we have $(clI)|Y=C(Y)$. It follows that

$$clI=\{f\in C(X): f=\text{constant on } E_\alpha \text{ for } {}^\forall\alpha,\ f=0 \text{ on } E_0\}$$

by a theorem of Bade and Curtis [2, p. 91, Proposition 1].

Proof of Theorem. Put

$$C_I(X)=\{f\in C(X): f=\text{constant on } E_\alpha \text{ for } {}^\forall\alpha\}$$

and

$$A_I=\{f\in A: f=\text{constant on } E_\alpha \text{ for } {}^\forall\alpha\}.$$

Then A_I is a closed subalgebra of A and $A_I\supset I$. Thus we may assume that A_I is a Banach function algebra on X/I. By Lemma 2 it is easy to see that $A+\bar{I}\supset C_I(X)$. It follows that $A_I+\bar{I}\supset C_I(X)$. For, if f is in $C_I(X)$ there are a g in A and an h in I such that $f=g+\bar{h}$. So $g=f-\bar{h}$ is constant on each E_α, that is, g is in A_I. Thus we have

$$A_I+\bar{A}_I\supset C_I(X).$$

By a Hoffman and Wermer and Bernard theorem [1], [4] on a uniformly closed real parts of a Banach function algebra, we see that $A_I=C_I(X)$. Thus I is uniformly closed since I is closed subalgebra of A_I by the definition of A_I. We conclude that

$$\begin{aligned} I&=clI\\ &=\{f\in C(X): f=\text{constant on } E_\alpha \text{ for } {}^\forall\alpha,\ f=0 \text{ on } E_0\}\\ &=cl\bar{I}=\bar{I}. \end{aligned}$$

Corollary 1. *Let A be a Banach function algebra on X and I be a closed subalgebra of A which separates the points of X. Suppose that $I\cdot A_R\subset I$ and* $\mathrm{Re}\,A\supset cl(\mathrm{Re}\,I)$. *Then we have $A=C(X)$ and $I=\{f\in C(X): f=0$ on $E_0\}$. Especially if I is a closed ideal of A such that $E_0=\phi$ or a one point set such that* $\mathrm{Re}\,A\supset cl(\mathrm{Re}\,I)$, *then we have $A=C(X)$ and $I=\{f\in C(X): f=0$ on $E_0\}$.*

Proof. Since I separates the points of X, it follows that $A+\bar{A}=C(X)$ by the same way as the proof of Theorem. Thus we see that $A=C(X)$ and $I=\{f\in C(X): f=0 \text{ on } E_0\}$.

Remark. In Corollary 1 the assumption "$I\cdot A_R\subset I$" is necessary. For example, put $A=\{f\in C(D): f(z) \text{ is analytic in } |z|<1/2\}$, where $D=\{z\in C: |z|\le 1\}$, and $I=\{f\in C(D): f \text{ is analytic on } |z|<1\}$ is the disk algebra on the closed unit disk. Then it is trivial that $I\subset A$ and I separates the points of X and $\mathrm{Re}\,A\supset cl(\mathrm{Re}\,I)$. On the other hand it is trivial that $A\neq C(X)$ and $A\neq I$.

Corollary 2. *Let A be a Banach function algebra on X and I be a closed ideal of A. Suppose that $A+\bar{I}\supset cl(I+\bar{I})$. Then we see $I=\bar{I}=\{f\in C(X): f=0$ on $E_0\}$.*

Proof. Each E_α is a one point set unless $E_\alpha=E_0$, since I is an ideal.

We can remove the hypothesis of the continuity of a functional $f\mapsto\bar{f}$ on $A\cap\bar{A}$ for a theorem of Mehta and Vasavada [5].

Corollary 3. *Let A be a Banach function algebra on X. Let N be a linear subspace of $C(X)$ and I be a closed ideal of A with $A+\bar{I}\supset N\supset I$. If $N+\bar{I}$ is uniformly closed, then $\bar{I}=I=\{f\in C(X): f=0$ on $E_0\}$.*

The author wishes to express his hearty thanks to Professor J. Wada for his valuable advice and constant encouragement.

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