46. Virtual Characters and Constant Coefficient Invariant Eigendistributions on a Semisimple Lie Group

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Introduction. Let G be a connected semisimple Lie group with finite centre. For simplicity, we assume that G is acceptable. Take an irreducible quasi-simple representation (π, E) of G on a Hilbert space E. Then we can define a distribution $\Theta(\pi)$ on G, called an *irreducible character* of (π, E), as follows:

$$\Theta(\pi)(f)=\mathrm{Trace}\int_G f(g)\pi(g)dg,$$

where f is a C^∞-function on G with compact support, and dg is a Haar measure on G. The notion of irreducible characters plays an important role in representation theory of such groups. For example, they decide irreducible representations of G up to infinitesimal equivalence.

Harish-Chandra studied the properties of irreducible characters during 1950's and 1960's. His study was very elaborate one and he got many deep results. However, most of the results follow from the fact that characters are invariant eigendistributions on G.

An *invariant eigendistribution* (IED) is a distribution on G which has the following two properties.

(1) It is invariant under inner automorphisms of G.

(2) It is a simultaneous eigendistribution of two-sided invariant differential operators on G.

As commented above, irreducible characters are IEDs. Then, it is natural to ask if characters and IEDs are different notions. As is well-known, when G is compact, they are essentially the same notion. When G is non-compact, this problem arised in 1970's. The first result was obtained by T. Hirai [3] in 1972 which says that they are really different. More precisely, there is a tempered IED which cannot be expressed as a linear combination of irreducible characters. Later, in 1974, Fomin and Shapovalov [1] proved that characters are constant coefficient IEDs (see Definition 1.2). In this article we report that a constant coefficient IED can be written as a linear combination of irreducible characters. Consequently, the space of constant coefficient IEDs and the linear span of irreducible characters with the same eigenvalue are identical (see Theorem 3.2).

§1. Constant coefficient IEDs. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_C$ its complexification. In this paper, we denote a Lie group by Roman capital letter and its Lie algebra by corresponding German small letter. Let $U(\mathfrak{g}_C)$ be the enveloping algebra of $\mathfrak{g}_C$ and $\mathfrak{Z}$ the centre of $U(\mathfrak{g}_C)$. Take $\chi \in \mathrm{Hom}_{alg}(\mathfrak{Z}, \boldsymbol{C})$. Then we denote by $\mathfrak{A}(\chi)$ (resp. $V(\chi)$) the space of IEDs (resp. the linear span of irreducible characters) with infinitesimal character χ. We say an IED Θ has an *infinitesimal character* χ if it satisfies $z\Theta=\chi(z)\Theta$ for any $z \in \mathfrak{Z}$. Then we have $V(\chi)\subset\mathfrak{A}(\chi)$.

One of the most important results on IEDs is the following theorem.

Theorem 1.1 (Harish-Chandra). *An IED is represented by a locally summable function on G which is real analytic on the set of regular elements G' of G.*

By this theorem we can determine an IED Θ by the values on the open dense subset G'. Let Car(G) be the set of conjugacy classes of Cartan subgroups in G. It is well-known that the number of conjugacy classes $\sharp$Car(G) is finite and

$$G'\subset \bigcup_{[H]\in \mathrm{Car}(G)} \bigcup_{g\in G} gHg^{-1},$$

where $[H]$ denotes the conjugacy class of a Cartan subgroup H. From the invariance of Θ, we see that Θ is completely determined by the values on a finite number of Cartan subgroups, a complete system of representatives of Car(G).

For a while, we fix a Cartan subgroup H of G. Put $W=W(\mathfrak{g}_C, \mathfrak{h}_C)$, the Weyl group of $(\mathfrak{g}_C, \mathfrak{h}_C)$. An IED $\Theta \in \mathfrak{A}(\chi)$ is written on H as follows. For a fixed $h\in H\cap G'$ and sufficiently small $x \in \mathfrak{h}$,

$$(D\Theta)(h \exp x)=\sum_{w\in W} c_w(h \exp x) \exp w\lambda(x),$$

where D is an analytic function on H called a *Weyl denominator* and $\lambda \in \mathfrak{h}_C^*$ is determined canonically by χ. In general, coefficients $c_w(h \exp x)$ is a polynominal function in x.

Definition 1.2. If the coefficients $c_w(h \exp x)$ are constants in x for any $w \in W$ and $[H] \in$ Car(G), we call Θ a *constant coefficient IED*. Denote the space of constant coefficient IEDs with infinitesimal character χ by $\mathfrak{A}'(\chi)$.

§2. Generalized principal series. Let H be a Cartan subgroup of G. Then there is a cuspidal parabolic subgroup P with Langlands decomposition $P=MAN$ satisfying the following. The reductive group M, which is not necessarily connected, contains a compact Cartan subgroup T and $H=TA$ is a direct product. Since M has a compact Cartan subgroup, M has discrete series representations and their limits.

Definition 2.1. Let σ be a (limit of) discrete series representation of M and ν a 1-dimensional character of A. By a *generalized principal series representation* induced from P, we mean the induced one, of the following type:

$$\pi(P\,;\sigma, \nu)=\pi(\sigma, \nu)=\mathrm{Ind}_{MAN}^G\, \sigma\otimes\nu\otimes 1_N,$$

where 1_N denotes the trivial representation of N.

The explicit formula of the character $\Theta(\pi(\sigma, \nu))$ of $\pi(\sigma, \nu)$ is well-known on the Cartan subgroup H ([2], [5]).

§ 3. **Main results and sketch of their proofs.** We get the following proposition.

Proposition 3.1. *A constant coefficient IED* $\Theta \in \mathfrak{A}'(\chi)$ *can be written as a linear combination of generalized principal series representations. That is, there exist triplets of cuspidal parabolics with Langlands decomposition* $P_i = M_i A_i N_i$, *(limits of) discrete series representations* σ_i *of* M_i *and 1-dimensional characters* ν_i *of* A_i *such that*

$$(*) \qquad \Theta = \sum_i c_i \Theta(\pi(P_i\,;\sigma_i, \nu_i)),$$

where c_i*'s are some constants.*

Once we establish this proposition, the following theorem is clear, taking the result of Fomin-Shapovalov into consideration. This is our main theorem.

Theorem 3.2. *The space of constant coefficient IEDs with infinitesimal character* χ *and the linear span of irreducible characters with infinitesimal character* χ *coincide with each other*, i.e., $\mathfrak{A}'(\chi) = V(\chi)$.

In the following, we sketch the proof of Proposition 3.1. At first, we remark that the explicit formulas of IEDs are known by the results of T. Hirai [4]. And we know the explicit formula of $\Theta(\pi(P\,;\sigma, \nu))$ on the associated Cartan subgroup H. So, we want to calculate the both hand sides of the equality $(*)$ on H. To do this, it is convenient to introduce an order on Car (G) (see [4, p. 273]).

For $[H_1], [H_2] \in \mathrm{Car}(G)$, we define $[H_1] < [H_2]$ if there is a sequence of Cartan subgroups $B_1, B_2, \cdots, B_n$ such that: (1) $[H_1]$ contains B_1 and $[H_2]$ contains B_n. (2) There is a Cayley transform c_α with respect to a real root α of $(\mathfrak{g}_C, (\mathfrak{b}_i)_C)$ such that $\mathfrak{b}_{i+1} = \mathfrak{g} \cap c_\alpha((\mathfrak{b}_i)_C)$ $(1 \leqq i \leqq n-1)$. We also define the *support* and the *height* of an IED Θ as follows.

$$\mathrm{Supp}\,(\Theta) = \{[H] \in \mathrm{Car}\,(G) \,|\, \Theta|_{H\cap G'} \neq 0\},$$

$$\mathrm{Hght}\,(\Theta) = \{[H] \in \mathrm{Supp}\,(\Theta) \,|\, [H] \text{ is maximal in } \mathrm{Supp}\,(\Theta)\}.$$

We call Θ *extremal* if Θ has the unique height.

By Hirai's results, any IED Θ is a linear combination of extremal ones, so we can assume Θ in Proposition 3.1 is extremal and of the height $[H]$. Corresponding to this H, we construct a cuspidal parabolic subgroup P as in § 2 and choose σ_j's and ν_j's so that we have the following identity on $H \cap G'$:

$$\Theta|_{H\cap G'} = \sum_j d_j \Theta(\pi(P\,;\sigma_j, \nu_j))|_{H\cap G'},$$

where d_j's are some constants.

Since Θ and $\Theta(\pi(P\,;\sigma_j, \nu_j))$'s are extremal IEDs of the height $[H]$, the IED $(\Theta - \sum d_j \Theta(\pi(P\,;\sigma_j, \nu_j)))$ has heights strictly less than $[H]$. Therefore we can complete the proof by induction on the order on Car (G).

The problem treated in this article arises in my work [7] which studies the Weyl group action on virtual character modules. We expect Weyl group action loosely classifies irreducible representations.

Recently, S. Mikami [6] studies when a given tempered IED can be written as a linear combination of "liftings" of stable IEDs in the sense of D. Shelstad. His result seems very interesting to us.

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References

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