5. *On the Spaces of Self Homotopy Equivalences for Fibre Spaces*

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1. Introduction. Throughout this paper, we shall work within the category of compactly generated spaces. A CW complex means a connected CW complex with base point which is a chosen vertex. Let X be a CW complex with base point x_0, $G(X)$ the space of self homotopy equivalences of X and $G_0(X)$ the space of self homotopy equivalences of (X, x_0). When X is an Eilenberg-MacLane complex $K(\pi, n)$, the weak homotopy type of $G(X)$ and $G_0(X)$ was determined by R. Thom [10] and D. H. Gottlieb [3]. We will denote $X \underset{w}{\simeq} Y$ if X has the same weak homotopy type as Y. In [11], [12], the author studied $G(X)$ and $G_0(X)$ when X is a certain product CW complex. The purpose of this paper is to study $G_0(X)$ when X is a fibre space of a Hurewicz fibration: $F \xrightarrow{i} X \xrightarrow{p} B$. In the sequel we simply call a Hurewicz fibration a fibration.

2. The function spaces and fibrations. When X and Y are spaces with base points x_0, y_0 we denote by map (X, Y) the space of maps of X to Y and by $\mathrm{map}_0(X, Y)$ the space of maps of (X, x_0) to (Y, y_0). Moreover, when k is a map of X to Y, we denote by map $(X, Y; k)$ the arcwise connected component of k in map (X, Y), and $\mathrm{map}_0(X, Y; k)$ is defined similarly.

Let X be a CW complex and A a subcomplex of X. And let $p: E \to B$ be a fibration. Then, we denote by $\mathrm{map}_0(X, B) \times' \mathrm{map}_0(A, E)$ the fibred product of the following fibrations $i^{\#}: \mathrm{map}_0(X, B) \to \mathrm{map}_0(A, B)$ and $p_{\#}: \mathrm{map}_0(A, E) \to \mathrm{map}_0(A, B)$, where $i^{\#}$ is induced by the inclusion $i: (A, x_0) \to (X, x_0)$ and $p_{\#}$ is induced by the projection $p: (E, e_0) \to (B, b_0)$. Now we define a map $\rho: \mathrm{map}_0(X, E) \to \mathrm{map}_0(X, B) \times' \mathrm{map}_0(A, E)$ by $\rho(f) = (p \circ f, f \circ i)$. Then we have the following propositions.

Proposition 1. $\rho: \mathrm{map}_0(X, E) \to \mathrm{map}_0(X, B) \times' \mathrm{map}_0(A, E)$ *is a fibration.*

Proposition 2. *Let E and B be CW complexes and let $p: E \to B$ be a fibration with fibre F which is a subcomplex of E. For a given $n > 1$, if F is $(n-1)$-connected and $\pi_i(B) = 0$ for every $i \geqq n$, then we have*

$$\mathrm{map}_0(E, B) \times' \mathrm{map}_0(F, E) \underset{w}{\simeq} \mathrm{map}_0(B, B) \times \mathrm{map}_0(F, F).$$

This proposition is proved by using the fact that $\mathrm{map}_0(F, B)$ is weakly contractible.

From Propositions 1 and 2 we get the first main result:

Theorem 1. *Under the hypothesis of Proposition 2, we have the following fibration:*

$$\mathcal{G}(E \bmod F) \longrightarrow G_0(E) \xrightarrow{\rho} B',$$

where B' is a subspace of $\mathrm{map}_0(E, B) \times' \mathrm{map}_0(F, E)$ with the same weak homotopy type as $G_0(B) \times G_0(F)$ and $\mathcal{G}(E \bmod F)$ is the space of self fibre homotopy equivalences of E leaving the fibre F fixed.

3. Fibration map theory. We introduce a fibration map theory for fibrations which corresponds to the bundle map theory for principal bundles initiated by I. M. James [6] and developed by D. H. Gottlieb [4], [5].

Let $p: E \to B$ and $p': E' \to B'$ be fibrations with arcwise connected base spaces B and B'. Then, let us begin with

Definition. Let $\tilde{f}: E \to E'$ and $f: B \to B'$ be maps such that $p' \circ \tilde{f} = f \circ p$. If $\tilde{f}$ carries each fibre of E into a fibre of E' by a homotopy equivalence, we call $\tilde{f}$ a *fibration map*.

Let $\mathcal{G}^*(E, E')$ be the space of fibration maps of E to E' and let $\mathcal{G}^*(E, E'; \tilde{h})$ be the arcwise connected component of $\tilde{h}$ in $\mathcal{G}^*(E, E')$. We define a map $\Phi: \mathcal{G}^*(E, E') \to \mathrm{map}(B, B')$ by setting $\Phi(\tilde{f}) = f$ for each fibration map $\tilde{f}: E \to E'$, where f is a map of B to B' induced by $\tilde{f}$. Then we have the fibration

$$\Phi: \mathcal{G}^*(E, E') \longrightarrow \mathrm{map}(B, B').$$

Moreover, let E, E', B and B' be CW complexes. Let B_0 be a connected subcomplex of B such that $p^{-1}(B_0) = E_0$ is a subcomplex of E. Then we have the following

Theorem 2. *Let $\tilde{i}: E_0 \to E$ be the inclusion, then*

$$\tilde{i}^{\#}: \mathcal{G}^*(E, E') \longrightarrow \mathcal{G}^*(E_0, E')$$

is a fibration.

Under the same situation as above, let furthermore $\tilde{\alpha}: E_0 \to E'$ be a fixed fibration map which induces a map $\alpha: B_0 \to B'$ and is extendable to a fibration map of E to E'. We denote by $\mathcal{G}^*_{\tilde{\alpha}}(E \bmod E_0, E')$ the space of fibration maps of E to E' which restrict on E_0 to $\tilde{\alpha}$. Let Φ denote also the restriction of $\Phi: \mathcal{G}^*(E, E') \to \mathrm{map}(B, B')$ on $\mathcal{G}^*_{\tilde{\alpha}}(E \bmod E_0, E')$. Then we have the following

Theorem 3. *This map*

$$\Phi: \mathcal{G}^*_{\tilde{\alpha}}(E \bmod E_0, E') \longrightarrow \mathrm{map}_\alpha(B \bmod B_0, B')$$

is a fibration, where $\mathrm{map}_\alpha(B \bmod B_0, B')$ is the space of maps from B to B' whose restriction on B_0 is the map α.

Let $\mathcal{G}(E)$ be the space of self fibre homotopy equivalences of E. Then we have the following theorems.

Theorem 4. *Let $p: E \to B$ and $p': E' \to B'$ be fibrations, where B and B' are CW complexes. A fibre $\Phi^{-1}(h)$ over h in the fibration*

$$\Phi: \mathcal{G}^*(E, E') \longrightarrow \mathrm{map}(B, B')$$

has the same homotopy type as $\mathcal{G}(E)$ for each h.

Theorem 5. *Under the hypothesis of Theorem 2, $\tilde{i}$ induces a map*

$$\tilde{i}^{\#}: \mathcal{G}(E) \longrightarrow \mathcal{G}(E_0),$$

which is a fibration.

Now, let $p: E \to B$ be a fibration with fibre F, where B and F are CW complexes. Then there exists a universal fibration $p_\infty: E_\infty \to B_\infty$ with fibre F, where B_∞ may be regarded as a classifying space $BG(F)$ [1], [7]. Then our preceding results (Theorems 2 and 3) together with the theorem of Gottlieb [4] yield

Theorem 6. *Under the hypothesis of Theorem* 2, $\mathcal{G}^*_{\tilde{k}\circ\tilde{i}}(E \bmod E_0, E_\infty; \tilde{k})$ *is weakly contractible, where* $\tilde{k}: E \to E_\infty$ *is a fibration map inducing a classifying map* $k: B \to B_\infty$ *for the fibration*: $F \longrightarrow E \xrightarrow{p} B$.

Let $\mathcal{G}(E \bmod E_0)$ be the space of self fibre homotopy equivalences of E leaving $p^{-1}(B_0) = E_0$ fixed, then from Theorem 6 we have the following

Theorem 7. *Under the hypothesis of Theorem* 2, *we have*

$$\mathcal{G}(E \bmod E_0) \underset{w}{\simeq} \Omega \operatorname{map}_{k\circ i}(B \bmod B_0, B_\infty; k),$$

where $k: B \to B_\infty$ *is a classifying map for the fibration* $p: E \to B$.

Remark. Our results on the fibration map theory have some overlaps with results of [2].

4. Applications. By using Theorem 7, we have the following

Theorem 8. *Let* $p: E \to B$ *be a fibration with fibre* $F = K(\pi, n)$ $(n > 1)$ *such that* E *and* B *are both* CW *complexes and* F *is a subcomplex of* E. *Assume* B *is simply connected, then we have*

$$\mathcal{G}(E \bmod F) \underset{w}{\simeq} \operatorname{map}_0(B, K(\pi, n)) \underset{w}{\simeq} H^n(B, \pi) \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi), i).$$

About the map $\rho: G_0(E) \to G_0(B) \times G_0(F)$, we have the following

Theorem 9. *Under the hypothesis of Proposition* 2, *the image of* ρ: $G_0(E) \to G_0(B) \times G_0(F)$ *is just the union of the arcwise connected components in* $G_0(B) \times G_0(F)$ *each of which contains* (g, h) *satisfying*

$$[\chi_\infty(h)] \circ [k] = [k] \circ [g],$$

where $\chi_\infty(h)$ *is a self map of* B_∞ *and* $k: B \to B_\infty$ *is a classifying map of the fibration* $p: E \to B$.

Let $\varepsilon(X)$ denote the group $\pi_0(G_0(X))$ for a CW complex X. By using Theorems 1, 8, and 9 we have

Theorem 10. *For given* $1 < m < n$, *let*

$$F = K(\pi', n) \xrightarrow{i} E \xrightarrow{p} K(\pi, m) = B$$

be a fibration with a classifying map $k: B \to K(\pi', n+1)$ $([k] \in H^{n+1}(B, \pi'))$ *and let* (E, F) *be a* CW *pair. Then we have*

$$G_0(E) \underset{w}{\simeq} R \times H^n(B, \pi') \times \prod_{i=1}^{n-1} K(H^{n-i}(B, \pi'), i),$$

where R *is the subgroup of* $\operatorname{Aut}(\pi) \times \operatorname{Aut}(\pi') = \varepsilon(B) \times \varepsilon(F)$ *consisting of* $([g], [h])$ *with*

$$(Bh)_*([k]) = g^*([k]).$$

Here g^* and $(Bh)_*$ are the automorphisms of $H^{n+1}(B, \pi')$ induced by g and Bh respectively.

As a corollary of Theorem 10, we have the following theorem proved by W. Shih [9] and Y. Nomura [8].

Theorem. *Under the hypothesis of Theorem* 10, *there exists the following exact sequence*

$$1 \longrightarrow H^n(B, \pi') \longrightarrow \varepsilon(E) \longrightarrow R \longrightarrow 1,$$

where R is the same group as the group stated in Theorem 10.

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