40. *On Homotopy Classes of Cochain Maps*

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1. Let R be a commutative ring. By a *cochain complex* over R, we mean a pair (C, δ) where

$$C=\sum_{n \in Z} C^n$$

is a graded R-module and $\delta: C \to C$ is a map of degree 1 such that $\delta^2=0$. We often abbreviate (C, δ) to C if there is no fear of confusion. Let (A, δ) and (B, δ') be cochain complexes over R. Then a *cochain map* from A to B is a graded module homomorphism $f: A \to B$ of degree 0 such that $\delta' f = f\delta$. Two cochain maps $f: A \to B$ and $g: A \to B$ are said to be *homotopic* if there exists a graded module homomorphism $\Delta: A \to B$ of degree -1 such that $\delta'\Delta+\Delta\delta=f-g$. In this case we write $f \simeq g$. We denote the abelian group of homotopy classes of cochain maps $A \to B$ by $[A, B]$. Let (M, δ) and (N, δ') be cochain complexes over R and $0 \to M \to X \to N \to 0$ be an exact sequence of cochain complexes over R. This exact sequence is called *weakly splitting* if the extension splits as that of graded modules, that is, the extension $0 \to M^n \to X^n \to N^n \to 0$ splits as the extension of R-modules for every dimension $n \in Z$. We say the weakly splitting extensions $0 \to M \to X \to N \to 0$ and $0 \to M \to X' \to N \to 0$ are *equivalent* if there exists a cochain map $\alpha: X \to X'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & X' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is commutative. Thus this equivalence relation is given by substituting the cochain maps for all the homomorphisms in the usual extensions of modules. We denote by $E_R(N, M)$ the set of all equivalence classes of weakly splitting extensions of N and M.

For a cochain map $f: A \to B$ the *mapping cone* $C(f)$ is defined to be $C(f)=A \oplus B$ (as graded modules) and $\delta(a, b)=(-\delta a,\ fa+\delta' b)$ (for every $a \in A^{n+1}$ and $b \in B^n$).

Then we have a weakly splitting extension of cochain complexes

$$(1) \qquad 0 \longrightarrow B \longrightarrow C(f) \longrightarrow A_\# \longrightarrow 0,$$

where $(A_\#, \delta_\#(f))$ is defined by

$$A_\#^n=A^{n+1}, \quad \delta_\#(f)a=-\delta a \quad (\text{for every } a \in A_\#^n).$$

We denote by $[f]$ the class of $[A, B]$ to which the cochain map $f: A \to B$ belongs. Let us denote the class of $E_R(A_\#, B)$ to which the extension (1) belongs by $\{C(f)\}$. The object of this note is to show an isomorphism

$$\Phi : [A, B] \simeq E_R(A_\#, B),$$

where Φ is defined by $\Phi([f]) = \{C(f)\}$.

2. Let $0 \to B \to X \to A_\# \to 0$ be any weakly splitting extension. By the fact that $X^n = A_\#^n \oplus B^n$ as R-module, the coboundary operator $\bar{\delta}$ of X is defined by $\bar{\delta}(a, b) = (-\delta a, fa + \delta' b)$, where $f = \sum_{n \in Z} f^n$ (f^n is a map from A^n to B^n). One can easily verify that $\bar{\delta}$ is a coboundary operator if and only if f is a cochain map. Hence every weakly splitting extension $0 \to B \to X \to A_\# \to 0$ can be identified with the extension $0 \to B \to C(f) \to A_\# \to 0$, where f is a cochain map defined by $(X, \bar{\delta})$ in the above way. Now we examine when the weakly splitting extensions are equivalent.

Suppose we are given two weakly splitting extensions

$$0 \longrightarrow B \longrightarrow C(f) \longrightarrow A_\# \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow B \longrightarrow C(g) \longrightarrow A_\# \longrightarrow 0, \tag{3}$$

where f and g are cochain maps from A to B. The extensions (2) and (3) are equivalent if and only if there exists a cochain map $\bar{\Delta}$ such that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C(f) & \longrightarrow & A_\# & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \bar{\Delta}} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & C(g) & \longrightarrow & A_\# & \longrightarrow & 0. \end{array}$$

The commutativity of this diagram implies $\bar{\Delta}(a, b) = (a, b + \Delta a)$, where Δ is a map of degree -1. If $\bar{\Delta}$ is a cochain map then Δ is a graded module homomorphism and $\bar{\Delta}\delta_\#(f) = \delta_\#(g)\bar{\Delta}$, which implies $(f - \Delta\delta)a = (g + \delta'\Delta)a$ for every $a \in A$. Hence $f - g = \delta'\Delta + \Delta\delta$. Conversely if $f \simeq g$, that is, $f - g = \delta'\Delta + \Delta\delta$ by some homotopy Δ, we see the extensions (2) and (3) are equivalent by the cochain map $\bar{\Delta}$. Thus the map Φ defined in §1 is shown to be bijective. Let us denote the Baer sum of $E_R(A_\#, B)$ by the symbol $*$. Then it is easy to see $\Phi([f] + [g]) = \Phi([f + g]) = \{C(f + g)\} = \{C(f)\} * \{C(g)\}$. Therefore Φ is an isomorphism of abelian groups. Thus we have proved the following theorem.

Theorem. *Let A and B are cochain complexes. $[A, B]$ denotes the abelian group of homotopy classes of cochain maps from A to B. $E_R(A_\#, B)$ denotes the abelian group of equivalent classes of weakly splitting extensions $A_\#$ and B. Then we have the isomorphism*

$$\Phi : [A, B] \simeq E_R(A_\#, B),$$

where $\Phi([f]) = \{C(f)\}$ for every $f \in \mathrm{Hom}(A, B)$.

References

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