39. *Polynomial Difference Equations which have Entire Solutions of Finite Order*

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1. Introduction. Here we consider the difference equation

(1.1) $$y(x+1)^m = a_p y(x)^p + a_{p-1} y(x)^{p-1} + \cdots + a_1 y(x) + a_0,$$

where $a_p, a_{p-1}, \cdots, a_1, a_0$ are constants, $a_p \neq 0$.

When $m=1$, the equation (1.1) has been studied by several authors [1], [4], [5]. We consider here mainly the case $m \geqq 2$.

We proved in [2] the following theorem.

Theorem A. *Let $R_j(x, w)$, $j=0, 1$, be rational functions:*

$$R_j(x, w) = P_j(x, w)/Q_j(x, w),$$
$$P_j(x, w) = a_{p_j}^{(j)}(x) w^{p_j} + \cdots + a_0^{(j)}(x),$$
$$Q_j(x, w) = b_{q_j}^{(j)}(x) w^{q_j} + \cdots + b_0^{(j)}(x),$$

in which $a_k^{(j)}(x)$ and $b_h^{(j)}(x)$, $k=0, \cdots, p_j$, $h=0, \cdots, q_j$, $j=0, 1$, are polynomials, $a_{p_j}^{(j)}(x) b_{q_j}^{(j)}(x) \not\equiv 0$. Consider the difference equation

(1.2) $$R_1(x, y(X+1)) = R_0(x, y(x)).$$

Suppose (1.2) *possesses a meromorphic solution $y(x)$, which is of finite order. Then, either $y(x)$ is rational, or there holds*

$$\max(p_1, q_1) = \max(p_0, q_0).$$

By this theorem, we know that the equation (1.1) admits a meromorphic solution of finite order only if

$$m = p.$$

In particular, when $m=1$, it is easy to see that (1.1) admits an entire solution of finite order if $p=1$. Our aim in this note is to determine the form of the equations (1.1) which have entire solutions of finite order, when $m \geqq 2$. Our results are as follows.

Theorem 1. *The equation* (1.1) *possesses an entire nontrivial solution of finite order if and only if it is either of the form*

(1.3) $$m \text{ is even and } y(x+1)^m = (A^2 - y(x)^2)^{m/2}, \ A \neq 0,$$

i.e.,

(1.3′) $$y(x+1)^2 = A^2 - y(x)^2,$$

or of the form

(1.4) $$y(x+1)^m = (ay(x)+b)^m.$$

By the way, we note that the equation (1.3) is satisfied by

$$y(x) = A \sin(\pi x/2) \quad \text{and} \quad y(x) = A \cos(\pi x/2).$$

The proof of Theorem 1 is implied in the following lemmas.

Lemma 2. *The equation* (1.1) *can not have an entire nontrivial solu-*

tion if $m=p\geqq 3$, *unless it is either of the form*

(1.5) $$y(x+1)^m=(ay(x)+b)^m,$$

or of the form

(1.5′) m *is even and* $y(x+1)^m=A(y(x)-c_1)^{m/2}(y(x)-c_2)^{m/2}$.

The equation of the form (1.5′) is nothing but the one with $m=2$.

Lemma 3. *Consider the equation* (1.1) *with* $m=p=2$, *which has an entire solution. If it is not of the form* (1.4), *then we have that*

$$a_2=-1 \quad \textit{and} \quad a_1=0.$$

2. Proof of Lemma 2. At first we remark that, if $y(x)$ is entire, then $y(x)$ can possess totally ramified values at most two [3, p. 277].

Suppose

(2.1) $$y(x+1)^m=A(y(x)-c_1)^{p_1}\cdots(y(x)-c_k)^{p_k}, \qquad c_j\neq c_h \text{ if } j\neq h,$$
$$p_1+\cdots+p_k=m.$$

Write $q_j=\text{G.C.D.}(m, p_j)$. By (2.1) we see that $y(x)$ is ramified over c_j to the order at least $m/q_j\geqq 2$. By the remark at the head of this section, we must have that $k\leqq 2$.

When $k=2$, we see by [3, p. 277] that

$$(1-q_1/m)+(1-q_2/m)\leqq 1, \qquad q_j/m\leqq 1/2.$$

Hence the equation (2.1) must be of the form (1.5′).

When $k=1$, the equation (2.1) is of the form (1.5). Q.E.D.

3. Proof of Lemma 3. Suppose

(3.1) $$y(x+1)^2=a_2y(x)^2+a_1y(x)+a_0$$
$$=a_2(y(x)-c_1)(y(x)-c_2).$$

If $c_1=c_2$, then (3.1) is of the form (1.4) with $m=2$.

If $c_1\neq c_2$, then

(3.2) $$a_0-a_1^2/(4a_2)=D\neq 0.$$

Suppose for an x_0

(3.3) $$y(x_0+1)^2=D.$$

Then by (3.1) we get $y(x_0)=-a_1/(2a_2)$. Differentiating (3.1), we obtain

(3.4) $$2a_2y(x)y'(x)+a_1y'(x)-2y(x+1)y'(x+1)=0.$$

By (3.3) and (3.4), we get $y'(x_0+1)=0$, since $y(x_0+1)^2=D\neq 0$. Therefore $y(x)$ is ramified over $\pm\sqrt{D}$. Since $y(x)$ is also ramified over c_1 and c_2, we must have that

$$(c_1, c_2)=(\sqrt{D}, -\sqrt{D}).$$

Hence we have

$$a_1=c_1+c_2=\sqrt{D}-\sqrt{D}=0.$$

Then

(3.5) $$y(x+1)^2=a_2y(x)^2+a_0.$$

By (3.5), we see that $y(x)$ is totally ramified over $\pm\sqrt{-a_0/a_2}$, and $y(x+1)$ is so over $\pm\sqrt{a_0}$. Therefore we get

$$-a_0/a_2=a_0, \quad \text{i.e.,} \quad a_2=-1,$$

which proves our lemma.

References

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