38. A Monotone Boundary Condition for a Domain with Many Tiny Spherical Holes

By Satoshi KAIZU
Department of Information Mathematics, University of Electro-Communications

(Communicated by Kôsaku YOSIDA, M. J. A., May 13, 1985)

1. Introduction. Let $\boldsymbol{R}^N$ be divided into an infinitely many number of cubes C_ε^i, $i\in N$, with volume of $(2\varepsilon)^N$. Let $B^i(r_\varepsilon)$ be a closed ball of radius $r_\varepsilon(<\varepsilon)$ set in the center of C_ε^i, here $N\geqq 3$. Let Ω be a bounded domain with smooth boundary Γ. We denote by F_ε the union of all balls $B^i(r_\varepsilon)$ $(\subset\Omega)$ such that dist $(B^i(r_\varepsilon),\Gamma)\geqq\varepsilon$. Let $\Omega_\varepsilon=\Omega\backslash F_\varepsilon$. Let ν be the outer unit normal of $\partial\Omega_\varepsilon$. For a positive number L_ε and a non-negative number c_ε we consider a monotone function β_ε defined by (i) $\beta_\varepsilon(r)=(r+c_\varepsilon)/L$ for $r\leqq -c_\varepsilon$, (ii) $\beta_\varepsilon(r)=0$ for $|r|\leqq c_\varepsilon$, (iii) $\beta_\varepsilon(r)=(r-c_\varepsilon)/L$ for $r\geqq c_\varepsilon$. In this paper we regard functions of $L^2(\Omega_\varepsilon)$ as functions of $L^2(\Omega)$ vanishing outside Ω_ε. For $f\in L^2(\Omega)$ we consider the boundary value problem:

(1) $$-\Delta u_\varepsilon=f \quad \text{a.e. in } \Omega_\varepsilon$$

(2) $$\frac{\partial u_\varepsilon}{\partial\nu}+\beta_\varepsilon(u_\varepsilon)=0 \quad \text{a.e. on } \partial\Omega_\varepsilon.$$

The problem admits a unique solution $u_\varepsilon\in H^2(\Omega_\varepsilon)$ (cf. [2]). We consider the behavior of u_ε under the condition

(3) $$\sup L_\varepsilon<\infty,\ c_\varepsilon\to 0,\ r_\varepsilon\to 0 \quad \text{and} \quad n_\varepsilon\to\infty$$

where n_ε is the number of holes of Ω_ε. Let $|\Omega|$ be the measure of Ω. In this paper the relation $n_\varepsilon\sim|\Omega|/(2\varepsilon)^N$ as $\varepsilon\to 0$ is very often used. Let b be a multivalued monotone function defined by (iv) the domain $D(b)=\{0\}$, (v) $b(0)=\boldsymbol{R}$. Replacing (2) by $\partial u_\varepsilon/\partial\nu+b(u_\varepsilon)\ni 0$ we obtain the Dirichlet boundary value problem.

The behavior of the Laplacian on domains with many tiny spherical holes, concerning the Dirichlet boundary condition, has been studied by M. Kac [3], J. Rauch and M. Taylor [6], S. Ozawa [5], D. Cioranescu and F. Murat [1] and other authors. Among them we shall extend the result of Cioranescu and Murat to the direction of the monotone boundary condition (2). Intuitively we have $\beta_\varepsilon\to b$ as $L_\varepsilon\to 0$ and $c_\varepsilon\to 0$. Thus the above idea may be natural. For another extension see S. Kaizu [4].

Theorem. *Let u_ε be the solution of* (1), (2) *and let $\tilde{u}_\varepsilon\in H^1(\Omega)$ be an extension of u_ε to be harmonic in F_ε. Take constants p, q such that $0\leqq p<\infty$ and $0\leqq q\leqq\infty$. We assume that the parameters r_ε, n_ε, c_ε and L_ε vary with* (3) *and*

(4) $$\sup c_\varepsilon/r_\varepsilon<\infty,\ n_\varepsilon r_\varepsilon^{N-2}\to p \quad \text{and} \quad L_\varepsilon/r_\varepsilon\to q.$$

Then $\tilde{u}_\varepsilon$ converges weakly in $H^1(\Omega)$ to the solution of

$$-\Delta u+\frac{(N-2)p|S_N|u}{(1+(N-2)q)|\Omega|}=f \qquad \text{a.e. } in\ \Omega,$$
$$u=0 \qquad \text{a.e. } on\ \Gamma,$$

where S_N *is the unit sphere of* $\boldsymbol{R}^N$ *and* $|S_N|$ *is the* $N-1$ *dimensional measure of* S_N. *Here we use the convention* $\infty^{-1}=0$.

2. **Proof of Theorem.** We assume that the parameters $\varepsilon_m, r_m, n_m, c_m$ and L_m satisfy (3), (4). We denote u_ε, Ω_ε and F_ε by u_m, Ω_m and F_m, respectively. Let $B_m=\cup\{B^i(\varepsilon_m): 1\leqq i\leqq n_m\}$. We denote by $M_0, M_1, \cdots$ generic constants independent of ε_m, r_m, c_m and L_m. We use the following property of β_m.

(5) $$[L_m\beta_m(r)]^2\leqq L_m\beta_m(r)r\leqq r^2.$$

We denote by $\tilde{v}\in H^1(\Omega)$ the extension of $v\in H^1(\Omega_m)$ to be harmonic in F_m. By an inequality in Example 1 of [6] we can see that there exists a constant M_0 such that

(6) $$\|\nabla\tilde{v}\|_{L^2(\Omega)^N}\leqq M_0\|\nabla v\|_{L^2(\Omega_m)^N}$$

for all $v\in H^1(\Omega_m)$ and all m. The variational formulation of (1), (2) is written as follows:

(7) $$\int_{\Omega_m}\nabla u_m\nabla v dx+\int_{\partial\Omega_m}\beta_m(u_m)v d\sigma=\int_{\Omega_m} f v dx$$

for all $v\in H^1(\Omega_m)$, where $\beta_m=\beta_{\varepsilon_m}$. Putting $v=u_m$ into (7), using (5) we obtain

(8) $$\|\nabla u_m\|^2_{L^2(\Omega_m)^N}+L_m^{-1}(\|U_m\|^2_{L^2(\partial\Omega_m)}+\|V_m\|^2_{L^2(\partial\Omega_m)})\leqq M_1\|\tilde{u}_m\|_{L^2(\Omega)}$$

with a certain constant M_1, where $U_m=0\vee(u_m-c_m)$ and $V_m=0\vee(-u_m-c_m)$; here U_m, $V_m\in H^1(\Omega_m)$. We write $\|v\|^2_{H^1(\Omega)}=\|\nabla v\|^2_{L^2(\Omega)^N}+\|v\|^2_{L^2(\Omega)}$. Using (6), (8) and the Poincaré inequality in $H^1(\Omega)$ we obtain

(9) $$\sup_m\|\tilde{u}_m\|_{H^1(\Omega)}<\infty,$$

(10) $$\sup_m(\|U_m\|^2_{L^2(\partial\Omega_m)}+\|V_m\|^2_{L^2(\partial\Omega_m)})/L_m<\infty.$$

Choose a subsequence still denoted by u_m such that $c_m\neq 0$ for all m and $\tilde{u}_m\to u$ weakly in $H^1(\Omega)$. Then $u\in H^1_0(\Omega)$ follows from (10). For the proof it suffices to show that u satisfies

(11) $$\int_\Omega\left[\nabla u\nabla\zeta+\frac{(N-2)p|S_N|u\zeta}{(1+(N-2)q)|\Omega|}\right]dx=\int_\Omega f\zeta dx$$

for all $\zeta\in C_0^\infty(\Omega)$. We shall modify Cioranescu and Murat's method applicable to our problem. We introduce $\{h_m\in W^{1,\infty}(\Omega_m)\}_m$ defined by (i) $h_m=1$ on $\Omega\backslash B_m$, (ii) $\Delta h_m=0$ on $B_m\backslash F_m$, (iii) $\partial h_m/\partial\nu+\beta_m(h_m)=0$ on ∂F_m. By direct calculations we see the concrete form of h_m on $B_m\backslash F_m$ (see Appendix). By this concrete form we see $\sup_m\|h_m\|_{H^1(\Omega)}<\infty$. By the same way as in [1] we have

(12) $$\begin{cases}\tilde{h}_m\xrightarrow{w}1 & \text{in } H^1(\Omega)\\ \dfrac{\partial h_m}{\partial r}\delta_m\xrightarrow{s}\dfrac{(N-2)p|S_N|}{(1+(N-2)q)|\Omega|} & \text{in } W^{-1,\infty}(\Omega),\end{cases}$$

where $\langle\delta_m, v\rangle=\int_{\partial F_m} v d\sigma$ for $v\in W_0^{1,1}(\Omega)$ and $\partial/\partial r$ is the outer normal derivative on the boundary ∂B_m of B_m. Set $I_m(v)=\int_{\partial F_m}[\beta_m(u_m)h_m-u_m\beta_m(h_m)]vd$ for

$v \in H^1(\Omega)$. For $w \in H^1(\Omega)$, putting $v = h_m w$ into (7) we obtain

$$I_m(w)=\int_\Omega [h_m(fw-\nabla\tilde{u}_m\nabla w)+u_m\nabla w\nabla\tilde{h}_m]\,dx-\int_{\partial B_m}\tilde{u}_m w\frac{\partial h_m}{\partial r}d\sigma \tag{13}$$
$$-\int_\Gamma \beta_m(u_m)w\,d\sigma.$$

Let $k_m=(h_m-c_m)|\partial F_m$. Let G_m^+ and G_m^- be the characteristic function of the sets $\{x \in \partial F_m : U_m>0\}$ and $\{x \in \partial F_m : V_m>0\}$, respectively. By the definition of β_m $I_m(\zeta)$ takes another form:

$$I_m(\zeta)=c_m L_m^{-1}\int_{\partial F_m}[(U_m-k_m)G_m^+ + (k_m-V_m)G_m^-]\zeta\,d\sigma \tag{14}$$
$$-k_m L_m^{-1}\int_{\partial F_m} u_m(1-G_m^+-G_m^-)\zeta\,d\sigma.$$

The relation

$$I_m(\zeta)\to 0 \quad \text{as } m\to\infty \tag{15}$$

follows from next two kinds of inequalities.

$$k_m L_m^{-1} \leqq M_2/r_m \quad\text{and}\quad |u_m(1-G_m^+-G_m^-)|\leqq c_m. \tag{16}$$

$$\sup_m (c_m/r_m)^{1/2}L_m^{-1}\max\left\{\int_{\partial F_m}U_m\,d\sigma,\ \int_{\partial F_m}V_m\,d\sigma\right\}<\infty. \tag{17}$$

We show the first half of (17). By (3) and (9) we have $\sup_m\|U_m\|_{H^1(\Omega)}<\infty$. Thus, by (5), (10), (12), (13) we have $\sup_m I_m(U_m)<\infty$. After replacing ζ by U_m in (14), using the Schwarz inequality to the first term of the right hand side of (14) and using the estimate $|\partial F_m|\leqq r_m/M_3$, $k_m L_m^{-1}\leqq M_2/r_m$, dividing both sides by L_m further, we get

$$M_4/L_m \geqq \left|(c_m/r_m)^{1/2}\int_{\partial F_m}U_m L_m^{-1}d\sigma\right|^2-(c_m/r_m)^{1/2}\int_{\partial F_m}U_m L_m^{-1}d\sigma \tag{18}$$

with a certain constant M_4. By (4), (10), the estimate on $|\partial F_m|$ and applying the Schwarz inequality on $\int_{\partial F_m}U_m\,d\sigma$, we see that, if $L_m\to 0$ with (4), then the value of the left hand side of (17) behaves similarly to the value of the second term of the right hand side of (18). Thus, the first half of (17) follows from (18). Similarly we obtain the remaining half of (17).

Lemma. *For* $\{v_m \in H^1(\Omega_m)\}_m$ *such that* $\sup_m\|v\|_{L^2(\partial F_m)}<\infty$ *we have*

$$\tilde{v}_m - v_m \xrightarrow{s} 0 \qquad \text{in } L^2(\Omega).$$

The sketch of the proof of Lemma is shown in [4]. By (10) and the concrete form of h_m Lemma is applicable to $\{h_m\}$, $\{u_m\}$. Then

$$\tilde{u}_m-u_m\xrightarrow{s}0 \text{ and } \tilde{h}_m-h_m\xrightarrow{s}0 \qquad \text{in } L^2(\Omega). \tag{19}$$

Using (12), (13) and (19) the proof of

$$I_m(\zeta)\to\int_\Omega\left[f\zeta-\nabla u\nabla\zeta-\frac{(N-2)p\,|S_N|\,u\zeta}{(1+(N-2)q)\,|\Omega|}\right]dx \qquad \text{as } m\to\infty \tag{20}$$

is done by the same way as in [4]. (11) follows from (15) and (20). q.e.d.

Appendix.

$$h_\varepsilon=\frac{L_\varepsilon(N-2)r_\varepsilon^{1-N}+(r_\varepsilon^{2-N}-\varepsilon^{2-N})-(1-c_\varepsilon)(r^{2-N}-\varepsilon^{2-N})}{L_\varepsilon(N-2)r_\varepsilon^{1-N}+r_\varepsilon^{2-N}-\varepsilon^{2-N}}.$$

References

[1] D. Cioranescu and F. Murat: Un terme étrange venu d'ailleurs I. Nonlinear partial differential equations and their applications. Res. Notes in Math., **60**, Pitman (1982).

[2] P. Grisvard: Smoothness of the solution of a monotonic boundary value problem for a second order elliptic equation in a general convex domain. Lect. Notes in Math., vol. 564, Springer (1977).

[3] M. Kac: Probabilistic methods in some problems of scattering theory. Rocky Mountain J. of Math., **4**, 511–538 (1974).

[4] S. Kaizu: The Robin problems on domains with many tiny holes. Proc. Japan Acad., **61**A, 39–42 (1985).

[5] S. Ozawa: On an elaboration of M. Kac's theorem concerning eigenvalues of the Laplacian in a region with randomly distributed small obstacles. Comm. Math. Physics, **91**, 473–487 (1983).

[6] J. Rauch and M. Taylor: Potential and scattering theory on wildly perturbed domains. J. Funct. Anal., **18**, 27–59 (1975).