35. A New Formulation of Local Boundary Value Problem in the Framework of Hyperfunctions. II

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This is a continuation of our previous paper [4]. In it we formulated non-characteristic boundary value problems for systems of linear partial differential equations and proved a Holmgren's type uniqueness theorem.

Here we first clarify the meaning of boundary values of hyperfunction solutions in the non-characteristic case by using F-mild hyperfunctions. Next we study boundary value problems for partial differential equations with regular singularities from our viewpoint apart from that of Kashiwara-Oshima [2]. Finally we microlocalize these boundary value problems in order to study micro-analyticity of solutions near the boundary.

We use the same notation as in [4]:

$$M=\boldsymbol{R}^n \ni x=(x_1, x'),\quad X=\boldsymbol{C}^n \ni z=(z_1, z'),\quad z'=(z_2, \cdots, z_n),$$
$$N=\{x \in M\,;\, x_1=0\},\quad Y=\{z \in X\,;\, z_1=0\},\quad \tilde{M}=\boldsymbol{R}\times\boldsymbol{C}^{n-1},$$
$$M_+=\{x \in M\,;\, x_1 \geqq 0\},\qquad \operatorname{int} M_+=\{x \in M\,;\, x_1>0\}.$$

We set $\mathcal{B}_{N|M_+}=(\iota_*\iota^{-1}\mathcal{B}_M)|_N$, where $\mathcal{B}_M$ is the sheaf of hyperfunctions on M and $\iota: \operatorname{int} M_+ \to M$ is the natural embedding.

§1. Non-characteristic boundary value problems. First let us recall the definition of F-mild hyperfunctions.

Definition 1 (Ôaku [5]). Let f be a germ of $\mathcal{B}_{N|M_+}$ at $\mathring{x} \in N$. Then f is called F-mild at $\mathring{x}$ if and only if f has a boundary value expression

$$f(x)=\sum_{j=1}^{J} F_j(x_1, x'+\sqrt{-1}\,\Gamma_j 0)$$

as a hyperfunction on $\{x \in \operatorname{int} M_+\,;\, |x-\mathring{x}|<\varepsilon\}$, where J is a positive integer, ε is a positive number, Γ_j is an open convex cone, F_j is a holomorphic function defined on a neighborhood (in $\boldsymbol{C}^n$) of

$$\{z=(z_1, z') \in \boldsymbol{C}^n\,;\, |z-\mathring{x}|<\varepsilon,\ \operatorname{Re} z_1 \geqq 0,\ \operatorname{Im} z_1=0,\ \operatorname{Im} z' \in \Gamma_j\}.$$

For an open set U of N, $\mathcal{B}^F_{N|M_+}(U)$ denotes the set of sections of $\mathcal{B}_{N|M_+}$ over U which are F-mild at each point of U. Then $\mathcal{B}^F_{N|M_+}$ is a subsheaf of $\mathcal{B}_{N|M_+}$ and called the sheaf of F-mild hyperfunctions. We denote by $\mathcal{D}_X$ the sheaf of rings of linear partial differential operators with holomorphic coefficients on X.

Theorem 1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is non-characteristic. Then we have*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}/\mathcal{B}^F_{N|M_+})=0,$$

and in particular

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^F_{N|M_+})=\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}).$$

Moreover the injective homomorphism

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

defined in Corollary of [4] *coincides with the one*

$$\gamma_0 : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^F_{N|M_+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N)$$

induced by the boundary value homomorphism $\mathcal{B}^F_{N|M_+} \to \mathcal{B}_N$ (*cf.* [5]); *here* $\mathcal{M}_Y = \mathcal{M}/z_1\mathcal{M}$ *is the tangential system of* $\mathcal{M}$ *to* Y.

Note that γ is invariant under real analytic local coordinate transformations of M preserving N and M_+ by virtue of this theorem and the invariance of $\mathcal{B}^F_{N|M_+}$ and γ_0. A result similar to Theorem 1 was proved by Kataoka [3] for single equations.

Sketch of the proof of Theorem 1. We set $\tilde{\mathcal{B}}^A = \mathcal{H}^{n-1}_N(\mathcal{O}_X|_Y)$, where $\mathcal{O}_X$ denotes the sheaf of holomorphic functions on X. Then there is an injective homomorphism $\beta : \tilde{\mathcal{B}}^A \to \tilde{\mathcal{B}}_{N|M_+}$ (cf. [4]). In view of the proof of Theorem 2 of [4], there are isomorphisms

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \xrightarrow{\sim} \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}),$$

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}^A) \xrightarrow{\sim} \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N).$$

These isomorphisms and the fact that α induces an injective homomorphism $\mathcal{B}_{N|M_+}/\mathcal{B}^F_{N|M_+} \to \tilde{\mathcal{B}}_{N|M_+}/\beta(\tilde{\mathcal{B}}^A)$ complete the proof.

§2. Boundary value problems for equations with regular singularities. We use the notation $D=(D_1, D')$, $D'=(D_2, \cdots, D_n)$ with $D_j = \partial/\partial z_j$. Let P be a linear partial differential operator with holomorphic coefficients defined on a neighborhood (in X) of $\mathring{x} \in N$. Suppose that P is written in the form

$$P = a(z)((z_1D_1)^m + A_1(z, D')(z_1D_1)^{m-1} + \cdots + A_m(z, D'))\,;$$

here $a(z)$ is a holomorphic function with $a(\mathring{x}) \neq 0$, and $A_j(z, D')$ is an operator of order $\leqq j$ free from D_1 such that $A_j(0, z', D')$ equals a function $a_j(z')$ (i.e. of order 0) for any $j=1, \cdots, m$. Then P is called an operator with regular singularities (in a weak sense) along Y after [2], or a Fuchsian operator of weight 0 after Baouendi-Goulaouic [1]. We denote by $\lambda = \lambda_1(z'), \cdots, \lambda_m(z')$ the roots of the indicial equation

$$\lambda^m + a_1(z')\lambda^{m-1} + \cdots + a_m(z') = 0.$$

These $\lambda_j(z')$ are called the characteristic exponents of P.

Theorem 2. *Let P be as above and assume, for any i, j, that $\lambda_i(\mathring{x}) - \lambda_j(\mathring{x})$ is not a nonzero integer. Set $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_XP$. Then on a neighborhood of $\mathring{x}$ there exists an injective sheaf homomorphism*

$$\gamma : \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M_+}) \longrightarrow (\mathcal{B}_N)^m.$$

In order to prove this theorem, we set $\Delta = \{(0, z', w') \in \{0\} \times \boldsymbol{C}^{n-1} \times \boldsymbol{C}^{n-1}\,;\ z' = w'\}$ and $dw' = dw_2 \wedge \cdots \wedge dw_n$.

Definition 2. $\mathcal{O}_0\tilde{\mathcal{D}} = \mathcal{H}^{n-1}_\Delta(\mathcal{O}_{\boldsymbol{C}^n \times \boldsymbol{C}^{n-1}}dw'|_{\{0\} \times \boldsymbol{C}^{n-1} \times \boldsymbol{C}^{n-1}})$.

Identifying Δ with Y, we regard $\mathcal{O}_0\tilde{\mathcal{D}}$ as a sheaf on Y. It is an extension ring of $\mathcal{O}_0\mathcal{D} = \{A \in \mathcal{D}_X|_Y\,;\ [z_1, A] = 0\}$ and acts on $\tilde{\mathcal{B}}_{N|M_+}$ (but not on $\mathcal{B}_{N|M_+}$).

Sketch of the proof of Theorem 2. By virtue of Theorem 1.3.6 of

Tahara [9] we can transform the equation $Pu=0$ into a system

$$\mathcal{N}: (z_1D_1+A_0(z'))v=0\,; \quad v=\begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}, \quad A_0=\begin{bmatrix} 0 & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & -1 \\ a_m & \cdots & & a_2 & a_1 \end{bmatrix}$$

making use of $\mathcal{O}_0\tilde{\mathcal{D}}$ (in fact, as is shown in [9], a suitable subring of $\mathcal{O}_0\tilde{\mathcal{D}}$ suffices). Hence we get

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{B}}_{N|M_+}) \cong \mathcal{H}om_{\mathcal{D}_X}(\mathcal{N}, \tilde{\mathcal{B}}_{N|M_+}) = \{x_1^{-A_0(x')}f(x')\,;\, f(x') \in (\mathcal{B}_N)^m\},$$

which, combined with α, proves the theorem.

Remark. Almost the same result as Theorem 2 was proved in [2] (see also Oshima [6]) in a less direct way, although we do not know if both definitions of boundary values coincide.

§3. Microlocalization. Let

$$\pi_{M|\tilde{M}} : (\tilde{M}\backslash M) \cup S^*_M\tilde{M} \longrightarrow \tilde{M}, \qquad \pi_{N|Y} : (Y\backslash N) \cup S^*_N Y \longrightarrow Y$$

be comonoidal transforms of $\tilde{M}$ and Y with centers M and N respectively (cf. [7]). Identifying $S^*_M\tilde{M} \underset{M}{\times} N$ with $S^*_N Y$, we set

$$\mathcal{C}_{M+} = \mathcal{H}^{n-1}_{S^*_M\tilde{M}}((\pi_{M|\tilde{M}})^{-1}\tilde{\imath}_*\tilde{\imath}^{-1}\mathcal{BO}_{\tilde{M}})^a, \qquad \mathcal{C}_{N|M+} = \mathcal{C}_{M+}|_{S^*_N Y},$$

$$\tilde{\mathcal{C}}_{N|M+} = \mathcal{H}^{n-1}_{S^*_N Y}((\pi_{N|Y})^{-1}\mathcal{BO}_{Y|\tilde{M}+})^a,$$

where a denotes the antipodal map (for $\tilde{\imath}$ and $\mathcal{BO}_{Y|\tilde{M}+}$ see [4]). In the same way as the theory of microfunctions (cf. [7]) we get exact sequences

$$0 \longrightarrow \mathcal{BO}_{Y|\tilde{M}+}|_N \longrightarrow \mathcal{B}_{N|M+} \longrightarrow (\pi_{N|Y})_*\mathcal{C}_{N|M+} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{BO}_{Y|\tilde{M}+}|_N \longrightarrow \tilde{\mathcal{B}}_{N|M+} \longrightarrow (\pi_{N|Y})_*\tilde{\mathcal{C}}_{N|M+} \longrightarrow 0.$$

We denote by $\mathcal{C}_M$ the sheaf on $S^*_M X$ of microfunctions. Since there exists an injective homomorphism of $\mathcal{C}_{N|M+}$ to $\tilde{\mathcal{C}}_{N|M+}$, we can prove the following by the same way as Corollary of [4] and Theorem 2.

Theorem 3. *Let $\mathcal{M}$ be as in Theorem 1 or 2. Then there exist an injective homomorphism*

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M+}) \longrightarrow \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

in case of Theorem 1, and an injective homomorphism

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M+}) \longrightarrow (\mathcal{C}_N)^m$$

defined on a neighborhood of $(\pi_{N|Y})^{-1}(\mathring{x})$ in case of Theorem 2. These homomorphisms are compatible with those in Theorems 1 and 2 respectively.

Let

$$p : S^*_M X \backslash S^*_{\tilde{M}} X \longrightarrow S^*_M \tilde{M} \quad \text{and} \quad \rho : S^*_M X \underset{M}{\times} N \backslash S^*_N M \longrightarrow S^*_N Y$$

be the canonical maps. There exists a sheaf homomorphism of $p^{-1}\mathcal{C}_{M+}$ to $\mathcal{C}_M$ on $(S^*_M X \underset{M}{\times} \operatorname{int} M_+)\backslash S^*_{\tilde{M}} X$. Hence we have the following microlocal version of Holmgren's theorem by virtue of Theorem 3.

Theorem 4. *Let $\mathcal{M}$ be as in Theorem 1 or 2 and f be a section of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_{N|M+})$ defined on a neighborhood of $\mathring{x} \in N$. Assume that $\gamma(f)$ is micro-analytic at $x^* \in (\pi_{N|Y})^{-1}(\mathring{x})$. Then $\rho^{-1}(x^*) \cup S^*_N M$ is disjoint from the closure of the singular spectrum of f regarded as a section of $\mathcal{H}om(\mathcal{M}, \mathcal{B}_M)$*

on $\{x \in \operatorname{int} M_+ ; |x-\mathring{x}|<\varepsilon\}$ *with an* $\varepsilon>0$.

This theorem was proved by Schapira [8] by a different method for single equations for which Y is non-characteristic.

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